ELSEVIER Journal of Geometry and Physics 25 (1998) 141–182

JOURNAL OF GEOMETRY AND PHYSICS

The local structure of n-Poisson and n-Jacobi manifolds ★

G. Marmo [a], G. Vilasi [b], A.M. Vinogradov [c,*],

[a] *Dipartimento di Scienze Fisiche, Università di Napoli, Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy*

[b] *Dipartimento di Scienze Fisiche E.R. Caianiello, Università di Salerno, Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Salerno, Italy*

[c] *Dipartimento di Ing. informatica e Matematica Appl., Università di Salerno, Istituto Nazionale di Fisica Nucleare, Gruppo Collegato di Salerno, Italy*

Received 7 February 1997

Abstract

n-Lie algebra structures on smooth function algebras given by means of multi-differential operators, are studied and their canonical forms are obtained.

Necessary and sufficient conditions for the sum and the wedge product of two n-Poisson structures to be again a multi-Poisson are found. It is proven that the canonical n-vector on the dual of an n-Lie algebra g is n-Poisson iff $\dim g \leq n+1$.

The problem of compatibility of two n-Lie algebra structures is analyzed and the compatibility relations connecting hereditary structures of a given n-Lie algebra are obtained. $(n+1)$-dimensional n-Lie algebras are classified and their "elementary particle-like" structure is discovered.

Some simple applications to dynamics are discussed. © 1998 Elsevier Science B.V.

Subj. Class.: Differential geometry
1991 MSC: 17B70, 58F05
Keywords: n-Lie algebra; n-Poisson (Nambu) bracket; n-Poisson (Nambu) manifold; n-Jacobi manifold

1. Introduction

The concept of n-Poisson structure (Nambu–Poisson manifold in terminology by Takhtajan) is a particular case of that of n-Lie algebra. To our knowledge the latter was introduced for the fist time by Filippov [7] in 1985 who gave first examples, developed first structural

★ Supported in part by the italian Ministero dell' Università della Ricerca Scientifica e Tecnologica.

* Corresponding author. Address: Facoltà di Scienze Matem. Fisiche e Naturali, Università degli studi di Salerno, I-84084 Fisciano, Italy. E-mail: vinograd@ponza.dia.unisa.it.

PII S0393-0440(97)00057-0

concepts, like simplicity, in this context and classified n-Lie algebras of dimensions $n+1$ which is parallel to the Bianchi classification of three-dimensional Lie algebras. Filippov defines an n-Lie algebra structure to be an n-ary multi-linear and anti-symmetric operation which satisfies the *n-ary Jacobi identity*

$$\begin{aligned}[][[u_1,\dots,u_n],v_1,\dots,v_{n-1}] = {} & [[u_1,v_1,\dots,v_{n-1}],u_2,\dots,u_n] \\ & + [u_1,[u_2,v_1,\dots,v_{n-1}],u_3,\dots,u_n] + \cdots \\ & + [u_1,\dots,u_{n-1},[u_n,v_1,\dots,v_{n-1}]]. \end{aligned} \tag{1.1}$$

Such an operation, realized on the smooth function algebra of a manifold and additionally assumed to be an n-derivation, is an *n-Poisson structure*. This general concept, however, was introduced neither by Filippov, nor, to our knowledge, by other mathematicians at that time. It was done much later in 1994 by Takhtajan [23] in order to formalize mathematically the n-ary generalization of Hamiltonian mechanics proposed by Nambu [20] in 1973. Apparently Nambu was motivated by some problems of quark dynamics and the n-bracket operation he considered was

$$\{f_1,\dots,f_n\} = \det\left\| \frac{\partial f_i}{\partial x_j} \right\|. \tag{1.2}$$

But Nambu himself as well as his followers do not mention that n-bracket (1.2) satisfies the n-Jacobi identity (1.1). On the other hand, Filippov reports (1.2) in his paper among other examples of n-Lie algebras. It seems that Filippov's work remained unnoticed by physicists. For instance, Takhtajan refers in [23] to a private communication by Flato and Fronsdal of 1992 who observed that the Nambu *canonical* bracket (1.2) satisfies the *fundamental identity* (1.1).

In this paper we study local n-Lie algebras, i.e. n-Lie algebra structures on smooth function algebras of smooth manifolds which are given by means of multi-differential operators. It follows from a theorem by Kirillov that these structure multi-differential operators are of the first order. We call *n-Jacobi* a local n-Lie algebra structure on a manifold. In the case when the structure multi-differential operator is a multi-derivation one gets an n-Poisson structure. So, n-Poisson manifolds form a subclass of n-Jacobi ones. The main mathematical result of the paper is a full local description of n-Jacobi and, in particular, of n-Poisson manifolds. This is an n-ary analog of the *Darboux lemma*. In what concerns n-Poisson manifolds the same result was also recently obtained by Alexeevsky and Guha [1]. Our approach is, however, quite different and, maybe, better reveals why n-Poisson and n-Jacobi structures reduce essentially to the functional determinants (1.2) (Theorems 4.1 and 5.1).

An important consequence of the n-Darboux lemma is that the cartesian product of two n-Jacobi, or two n-Poisson manifolds does not give a manifold of the same type if $n > 2$. Possibly this fact may explain the remarkable inseparability of quarks. This possibility suggests to investigate better the relevance of local n-Lie algebra structures for quark dynamics. The structure of $(n+1)$-dimensional n-Lie algebras which is described in Section 6 seems to be in favor of such idea.

It was not our unique goal in this paper to describe local structure of local n-Lie algebras. First, we tried to be systematic in what concerns the relevant basic formulae and constructions. Second, possible applications of the developed theory to integrable systems and related problems of dynamics are illustrated on some examples of current interest.

More precisely, the content of the paper is as follows.

In Section 2 the necessary generalities concerning n-Lie algebras and their derivations are reported. A new point discussed there is the concept of *compatibility* of two n-Lie structures defined on the same vector space. Two compatible structures can be combined to get a third one. This is why this concept seems to be of a crucial importance even for the theory of usual, i.e. 2-Lie, algebras. Fixing a number of arguments in an n-Lie bracket one gets new multi-linear Lie algebras of lower multiplicities, called *hereditary*. We deduce the compatibility relations tacking together hereditary structures of a given n-Lie algebra.

Generalities on n-Poisson manifolds are collected in Section 3. There we introduce and discuss such basic notions related to an n-Poisson manifold as the *Casimir algebra*, *Casimir map* and *Hamiltonian foliation*. It is shown that n-Poisson structures allow for multiplication by smooth functions if $n \geq 3$.

The main structure result regarding n-Poisson structures (Theorem 4.1) is proved in Section 4. It tells that the structure n-vector of an n-Poisson structure is of rank n (*decomposable*) if $n > 2$. This leads directly to the *n-Darboux lemma*: *Given an n-Poisson structure, $n > 2$, on a manifold M there exists a local chart $x_1, \ldots, x_m$, $m = \dim M \geq n$, on M such that the corresponding n-Poisson bracket is given by* (1.2). Two consequences of this result are worth mentioning. First, the n-bracket defined naturally on the dual of an n-Lie algebra $\mathcal{V}$ is not generally an n-Poisson structure if $n > 2$. This is in sharp contrast with usual, i.e. $n = 2$, Lie algebras. However, we show that it is still so for n-dimensional and $(n+1)$-dimensional n-Lie algebras. By this and some other reasons it is natural to conjecture that n-Lie algebras with $n > 2$ are essentially n-dimensional and $(n+1)$-dimensional ones. Finally, in this section we deduce necessary and sufficient conditions in order the wedge product of two multi-Poisson structures be again a Poisson one.

The *n-Darboux* lemma for general n-Jacobi manifolds with $n > 2$ is proved in Section 5, Theorem 5.1 and Corollary 5.7. The key idea in doing that is to split a first order multi-differential operator into two parts similar to the canonical representation of a scalar first order differential operator as the sum of a derivation and a function. An n-ary analog of the well-known Bianchi classification of three-dimensional Lie algebras is given in Section 6. An exhaustive description of $(n+1)$-dimensional n-Lie algebras was already done by Filippov [7] by a direct algebraic approach. Our approach is absolutely different and based on the use of the natural n-Poisson structure on the dual of an $(n+1)$-dimensional n-Lie algebra. It allows to get the classification in a very simple and transparent way and, what is more important, to discover what we would like to call an *elementary particle-like* structure of $(n+1)$-dimensional n-Lie algebras. More exactly, we show that any such algebra is a specific linear combination of two *simplest* n-Lie algebra types realized in a mutually compatible (in the sense of Section 2) way. A number similar to the coupling constant appears in this context. In this section we describe also derivations of $(n+1)$-dimensional n-Lie algebras and realize the Witt (or $sl(2, \mathbb{R})$–Kac–Moody) algebra as a 2-Lie subalgebra of the canonical 3-algebra structure on $\mathbb{R}^3$. In the concluding Section 7 we exhibit on concrete examples some simple applications of n-ary structures to dynamics. First, we use the Kepler dynamics to show how the constants of motion can be put in relation with multi-Poisson structures. Second, alternative Poisson realizations of a spinning particle dynamics Γ are given by using ternary structures preserved by Γ. In a separate paper applications to dynamics of the developed formalism will be discussed more systematically.

The multi-generalization of the concept of (local) Lie algebra studied in this paper is not, in fact, unique and there are other natural alternatives (see [9,19,15,27]). All these generalizations are mutually interrelated and open very promising perspectives for particle and field dynamics.

In this article we follow Filippov in what concerns the terminology and use *n-Lie algebra* instead of Takhtajian's *Nambu–Lie gebras*. The reason is that arabic *al-gebre* became ethymologically indivisible in the current mathematical language , like *ring*, *group*, etc. So, it would be hardly convenient to use *n-gebra* together with the indisputable *n-ring*.

2. n-Lie algebras

We start with some basic definitions.

Definition 2.1. An n-Lie algebra structure on a vector space $\mathcal{V}$ (over a field $\mathbf{K}$) is a multilinear mapping of $\mathcal{V} \times \cdots \times \mathcal{V}(n \ \ times)$ to $\mathcal{V}$ such that for any $u_i, v_j \in \mathcal{V}$, the n-Jacobi identity (1.1) holds.

Remark 2.1. It is convenient to treat the ground field $\mathbf{K}$ as the unique 0-Lie algebra and a linear space supplied with a linear operator as a 1-Lie algebra.

If an n-Lie algebra is fixed in the current context we refer to the underlying vector space $\mathcal{V}$ as the n-Lie algebra in question (as it is common for the *usual* Lie algebras). However, sometimes we need consider two or more n-Lie algebra structures on the same vector space. In such a situation we use $P(u_1, \ldots, u_n)$ instead of $[u_1, \ldots, u_n]$. This notation appeals directly to the n-Lie algebra in question and is more flexible than the use of alternative bracket graphics.

Example 2.1 [7]. Let $\mathcal{V}$ be an $(n+1)$-dimensional vector space over $\mathbb{R}$ supplied with an orientation and a scalar product $(\cdot, \cdot)$.

The *n-vector product* $[v_1, \ldots, v_n]$ of $v_1, \ldots, v_n \in \mathcal{V}$ is defined uniquely by requirements:

1. $[v_1, \ldots, v_n]$ is orthogonal to all v_i's;
2. $|[v_1, \ldots, v_n]| = \det \|(v_i, v_j)\|^{1/2}$;
3. the ordered system $v_1, \ldots, v_n, [v_1, \ldots, v_n]$ conforms the orientation of $\mathcal{V}$.

Let P and Q be n-Lie algebra structures on $\mathcal{V}$ and $\mathcal{W}$, respectively. Then their direct product $R = P \oplus Q$ defined as

$$R((v_1, w_1), \ldots, (v_n, w_n)) = (P(v_1, \ldots, v_n), Q(w_1, \ldots, w_n))$$

with $v_i \in \mathcal{V}, \ w_i \in \mathcal{W}$ is an n-Lie algebra structure on $\mathcal{V} \oplus \mathcal{W}$.

A central notion in the theory of n-Lie algebras is that of derivation [7].

Definition 2.2. A linear map $\mathcal{D} : \mathcal{V} \to \mathcal{V}$ is said to be a derivation of the n-Lie algebra $\mathcal{V}$ if for any $u_1, \ldots, u_n \in \mathcal{V}$

$$\mathcal{D}[u_1, \ldots, u_n] = \sum_{i=1} [u_1, \ldots, \mathcal{D}u_i, \ldots, u_n]. \tag{2.1}$$

Fixing arbitrary elements $u_1, \ldots, u_{n-1} \in \mathcal{V}$ one gets a map $v \to [u_1, \ldots u_{n-1}, v]$ which is a derivation of $\mathcal{V}$ as it follows from the Jacobi identity (1.1). Such a derivation is called *pure inner associated with* $u_1, \ldots, u_{n-1}$. It will be denoted by $ad_{u_1,\ldots,u_{n-1}}$ or $P_{u_1,\ldots,u_{n-1}}$ for the n-Lie algebra structure P in question. Linear combinations of pure inner derivations will be called *inner derivations (of P)*. Note that the concepts of inner and pure inner coincide for $n = 2$ and that Hamiltonian vector fields are inner derivations of the background Poisson structure. Following the standard terminology we, sometimes, shall call *outer*, derivations of $\mathcal{V}$ which are not inner just to stress the instance of it.

Proposition 2.1. *Derivations of an n-Lie algebra form a Lie algebra with respect to the standard commutation operation and inner derivations constitute an ideal of it.*

Proof. Let $\mathcal{D}_1, \mathcal{D}_2$ be derivations of the bracket $[\cdot, \ldots, \cdot]$. Then, obviously,

$$\begin{aligned}\mathcal{D}_1(\mathcal{D}_2([u_1, \ldots, u_n])) = \sum_{i<j} ([\ldots, \mathcal{D}_1 u_i, \ldots, \mathcal{D}_2 u_j, \ldots] \\ + [\ldots, \mathcal{D}_2 u_i, \ldots, \mathcal{D}_1 u_j, \ldots]) \\ + \sum_i [u_1, \ldots, \mathcal{D}_1\mathcal{D}_2 u_i, \ldots, u_n].\end{aligned} \tag{2.2}$$

Therefore,

$$[\mathcal{D}_1, \mathcal{D}_2]([u_1, \ldots, u_n]) = \sum_i [u_1, \ldots, [\mathcal{D}_1, \mathcal{D}_2]u_i, \ldots, u_n]. \tag{2.3}$$

The first assertion in the proposition is so proven. The second assertion follows by observing that for a derivation $\mathcal{D}$

$$\begin{aligned}[\mathcal{D}, ad_{u_1,\ldots,u_{n-1}}]u &= \mathcal{D}([u_1, \ldots, u_{n-1}, u]) - [u_1, \ldots, u_{n-1}, \mathcal{D}u] \\ &= \sum_{i \le n-1} [u_1, \ldots, \mathcal{D}u_i, \ldots u_{n-1}, u]\end{aligned}$$

So, in virtue of (2.3) one has

$$
\begin{aligned}
&[\mathcal{D}, ad_{u_1,\dots u_{n-1}}]([v_1,\dots,v_n])\\
&\quad = \sum_i [v_1,\dots,[\mathcal{D}, ad_{u_1,\dots,u_{n-1}}]v_i,\dots,v_n]\\
&\quad = \sum_i \left(\sum_{s\le n-1} [v_1,\dots,[u_1,\dots,\mathcal{D}u_s,\dots u_{n-1},v_i],\dots,v_n]\right)\\
&\quad = \sum_{s\le n-1} \left(\sum_i [v_1,\dots,[u_1,\dots,\mathcal{D}u_s,\dots u_{n-1},v_i],\dots,v_n]\right)\\
&\quad = \sum_{s\le n-1} ad_{u_1,\dots,\mathcal{D}u_s,\dots,u_{n-1}}([v_1,\dots,v_n])
\end{aligned} \tag{2.4}
$$

In other words,

$$[\mathcal{D}, ad_{u_1,\dots u_{n-1}}] = \sum_{s\le n-1} ad_{u_1,\dots,\mathcal{D}u_s,\dots,u_{n-1}} \tag{2.5}$$

or, with the alternative notation

$$[\mathcal{D}, P_{u_1,\dots u_{n-1}}] = \sum_{s\le n-1} P_{u_1,\dots,\mathcal{D}u_s,\dots,u_{n-1}}. \quad \square \tag{2.6}$$

By putting $\mathcal{D} = P_{v_1,\dots v_{n-1}}$ in (2.6) one gets the commutation formula for pure inner derivations

$$[P_{v_1,\dots,v_{n-1}}, P_{u_1,\dots,u_{n-1}}] = \sum_i P_{u_1,\dots,[v_1,\dots,v_{n-1},u_i],\dots,u_{n-1}}. \tag{2.7}$$

Note also the following relation in the algebra of inner derivations of P which is due to skew-commutativity of the left-hand side commutator in (2.7):

$$\sum_i P_{u_1,\dots,[v_1,\dots v_{n-1},u_i],\dots,u_{n-1}} + \sum_i P_{v_1,\dots,[u_1,\dots,u_{n-1},v_i],\dots,v_{n-1}} = 0.$$

A description of the derivation algebra of an $(n+1)$-dimensional n-Lie algebra is given in Proposition 6.6, see also [7]. Various outer derivations of an "atomic" four-dimensional 3-Lie algebra are presented in Example 6.1.

While the above results are just straightforward generalizations of known elementary facts of the standard Lie algebra theory the following simple observation (due to Filippov) is a very important new peculiarity of n-ary Lie algebras with $n > 2$.

Proposition 2.2. *Let P be an n-Lie algebra structure on $\mathcal{V}$. Then for any $u_1,\dots,u_k \in \mathcal{V}$, $k \le n$, $P_{u_1,\dots,u_k}$ is an $(n-k)$-Lie algebra structure on $\mathcal{V}$.*

Proof. It is sufficient, obviously, to prove this result for $k = 1$ only. But in this case one can see easily that the Jacobi identity for P_u is obtained from that of P just by putting in it $u_n = u_{n-1} = u$. $\square$

Example 2.2. If P is the n-vector product structure of Example 2.1, then the $(n-k)$-Lie algebra structure $P_{u_1,\dots,u_k}$ on $\mathcal{V}$ is the direct product of the trivial structure on $S = Span\{u_1,\dots,u_k\}$ and the $(n-k)$-vector product structure on $S^\perp$ with respect to the scalar product

$$(\cdot,\cdot)' = \lambda(\cdot,\cdot)|_{S^\perp}, \qquad \lambda = (vol_k(u_1,\dots,u_k))^{1/(n-k)},$$

on $S^\perp$.

Multi-Lie structures $P_{u_1,\dots,u_k}$ obtained in this way from P will be called *hereditary* (with respect to P) of order k. The fact that these structures belong to the same *family* implies mutual *compatibility* of them, an important concept we are going to discuss.

With this purpose we need first the following analog of the Lie derivation operator. Let $Q : \mathcal{V}\times\cdots\times\mathcal{V}\to\mathcal{V}$ be a k-linear mapping and $\partial : \mathcal{V}\to\mathcal{V}$ be a linear operator. The ∂-derivative $\partial(Q)$ of Q is also a k-linear map defined as

$$[\partial(Q)](u_1,\dots,u_k) = \partial(Q(u_1,\dots,u_k)) - \sum_i Q(u_1,\dots,\partial u_i,\dots,u_k).$$

Note that the Jacobi identity (1.1) is equivalent to $P_{u_1,\dots,u_k}(P) = 0$ for any $u_1,\dots,u_k\in\mathcal{V}$.

Example 2.3. If $k = 1$, i.e. Q is a linear operator on $\mathcal{V}$, then $\partial(Q) = [\partial, Q]$.

Sometimes it is more convenient to use L_∂ instead of ∂ for the ∂-derivative. An instance of it is the formula

$$[L_\partial, \iota_u] = \iota_{\partial(u)}, \tag{2.8}$$

where ι_u for $u\in\mathcal{V}$ denotes the insertion operator, i.e.

$$\iota_u(Q)(u_1,\dots,u_{k-1}) = Q(u,u_1,\dots,u_{k-1}). \tag{2.9}$$

The proof of (2.8) is trivial.

Definition 2.3. Two n-Lie algebra structures on $\mathcal{V}$ are said to be compatible if for any $u_1,\dots,u_{n-1}\in\mathcal{V}$

$$P_{u_1,\dots,u_{n-1}}(Q) + Q_{u_1,\dots,u_{n-1}}(P) = 0. \tag{2.10}$$

Remark 2.2. If $\mathcal{V} = C^\infty(M)$, $n = 2$ and P and Q are two Poisson structures on M, then they are compatible in the well-known sense of Magri [17] (see also [5,6,13]) iff they are compatible in the sense of Definition 2.3. It is not difficult to see that in such a situation condition (2.10) is identical to the vanishing of the Schouten bracket of P and Q.

Example 2.4. For $n = 1$ the compatibility condition is empty. In fact, in this case P and Q are just linear operators and

$$P(Q) + Q(P) = [P, Q] + [Q, P] = 0.$$

The following proposition gives a possible interpretation of the notion of compatibility.

Proposition 2.3. *Let P and Q be n-Lie structures on V. If $a, b \in \mathbf{K}$, $ab \neq 0$, then $aP+bQ$ is an n-Lie algebra structure iff P and Q are compatible.*

Proof. The following identity is due to linearity of the *Lie derivative* expression $I(R)$ with respect to both I and R:

$$(aP+bQ)_{u_1,\dots,u_{n-1}}(aP+bQ) = a^2P_{u_1,\dots,u_{n-1}}(P) + abP_{u_1,\dots,u_{n-1}}(Q) + abQ_{u_1,\dots,u_{n-1}}(P) + b^2Q_{u_1,\dots,u_{n-1}}(Q).$$

It remains now to apply interpretation (2.8) of the Jacobi identity. □

Example 2.5. Let $\mathcal{A}$ be an associative algebra. For a given $M \in \mathcal{A}$ define a skew-symmetric bracket $[\cdot\,,\,\cdot]_M$ on $\mathcal{A}$ by putting

$$[A, B]_M = AMB - BMA, \quad A, B \in \mathcal{A}. \tag{2.11}$$

It is easy to see that this, in fact, is a Lie algebra structure on $\mathcal{A}$. Moreover, for any $M, N \in \mathcal{A}$ structures $[\cdot\,,\,\cdot]_M$ and $[\cdot\,,\,\cdot]_N$ are compatible. This follows from the fact that

$$[\cdot\,,\,\cdot]_M + [\cdot\,,\,\cdot]_N = [\cdot\,,\,\cdot]_{M+N}.$$

Corollary 2.1. *Any two first order hereditary structures P_u and P_v of an n-Lie algebra P are compatible.*

Proof. In fact, according to Proposition 2.2, $P_u + P_v = P_{u+v}$ is an $(n-1)$-algebra structure. □

On the contrary, hereditary structures of an order greater than 1 are not, in general, mutually compatible. It can be seen as follows.

Denote by $Comp(P, Q; u_1, \dots, u_{n-1})$ the left-hand side of the compatibility condition (2.10). Then a direct computation shows that

$$Comp(P_{u,v}, P_{w,z}; u_1, \dots, u_{n-3}) = P_{P(u,v,u_1,\dots,u_{n-3},w),z} + P_{w,P(u,v,u_1,\dots,u_{n-3},z)} + P_{P(w,z,u_1,\dots,u_{n-3},u),v} + P_{u,P(w,z,u_1,\dots,u_{n-3},v)}.$$

In particular, for $u_1 = u$ we have

$$Comp\,(P_{u,v}, P_{w,z}; u, u_2, \dots, u_{n-3}) = P_{u,P(w,z,u,u_2\dots,u_{n-3},v)} = Q_{Q(w,z,u_2\dots,u_{n-3},v)}$$

with $Q = P_u$. Now one can see from an example that $Q_{Q(w,z,u_2\dots,u_{n-3},v)}$ is generically different from zero. For instance, if P is the n-vector product algebra, then $Q = P_u$ is isomorphic to the direct sum of the $(n-1)$-vector product algebra and the trivial one-dimensional one. Then $Q_{Q(w,z,u_2,\dots,u_{n-3},v)} \neq 0$ for linearly independent $w, z, u_2, \dots, u_{n-3}, v$ belonging to the first direct summand. However, second order hereditary structures are subjected to another kind of relations deriving from that of compatibility. To describe them it will be convenient to introduce a symmetric bilinear function $Comp(P, Q)$ defined by

$$Comp(P, Q)(u_1, \dots, u_{n-1}) = Comp(P, Q; u_1, \dots, u_{n-1}). \tag{2.12}$$

By definition Comp(P, Q) is an $(n-1)$-linear skew-symmetric function on $\mathcal{V}$ with values in the space of n-linear skew-symmetric functions on $\mathcal{V}$. By this reason we have, in particular,

$$\begin{aligned} Comp(P_{u+w,v}, P_{u+w,z}) &= Comp(P_{u,v}, P_{u,z}) + Comp(P_{u,v}, P_{w,z}) \\ &\quad + Comp(P_{w,v}, P_{u,z}) + Comp(P_{w,v}, P_{w,z}). \end{aligned}$$

Note now that two second order hereditary structures of the form $P_{x,y}$, $P_{x,z}$ are compatible because they can be regarded as first order hereditary structures of the $(n-1)$-Lie algebra P_x. By this reason the above equality reduces to

$$Comp(P_{u,v}, P_{w,z}) + Comp(P_{u,z}, P_{w,v}) = 0. \tag{2.13}$$

Identity (2.13) binding second order secondary structures tells that the compatibility condition between $P_{u,v}$ and $P_{w,z}$ depends rather on bivectors $u \wedge v$ and $w \wedge z$ than on vectors u, v and w, z representing them, correspondingly.

Similar relations binding together kth order hereditary structures can be found by generalizing properly the above reasoning. With this purpose we need to develop a suitable notation associated with a fixed n-Lie algebra structure P on $\mathcal{V}$. Let $v_1, \dots, v_k, w_1, \dots, w_k \in \mathcal{V}$, $i = 1, \dots, k$.

Let us define the symbol $\langle v_1, \dots, v_k | w_1, \dots, w_k \rangle$ by

$$\begin{aligned} &\langle v_1, \dots, v_k | w_1, \dots, w_k \rangle (u_1, \dots, u_{n-k-1}) \\ &\quad = Comp(P_{v_1,\dots,v_k}, P_{w_1,\dots,w_k}; u_1, \dots, u_{n-k-1}). \end{aligned}$$

So, $\langle v_1, \dots, v_k | w_1, \dots, w_k \rangle$ is a skew-symmetric $(n-k-1)$-linear function on $\mathcal{V}$ with values in the space of $(n-k)$-linear skew-symmetric functions on $\mathcal{V}$. Moreover, it is symmetric with respect to v and w, i.e.

$$\langle v_1, \dots, v_k | w_1, \dots, w_k \rangle = \langle w_1, \dots, w_k | v_1, \dots, v_k \rangle \tag{2.14}$$

and skew-symmetric with respect to variables v_i's as well as w_i's. If $I = (i_1, \dots, i_p)$ is a sequence of integers such that $i_1 < \cdots < i_p$, then $(v, w)_I$ stands for the sequence of n elements of $\mathcal{V}$ such that its sth term is v_s if $s \in I$ and w_s otherwise. A similar meaning has the symbol $(w, v)_I$. For example, if $k = 5$ and $I = (1, 3)$, then $(v, w)_I = (v_1, w_2, v_3, w_4, w_5)$, $(w, v)_I = (w_1, v_2, w_3, v_4, v_5)$. Define now the following *quadratic* function:

$$C(v_1, \dots, v_k | w_1, \dots, w_k) = \sum_{I, i_1 = 1} \langle (v, w)_I | (w, v)_I \rangle. \tag{2.15}$$

Proposition 2.4. *For any $v_1, \dots, v_k, w_1, \dots, w_k \in \mathcal{V}$, $n \geq k$, it holds*

$$C(v_1, \dots, v_k | w_1, \dots, w_k) = 0. \tag{2.16}$$

Equality (2.16) is called *the kth order compatibility condition.*

Remark 2.3. Corollary 2.1 is identical to (2.16) for $k = 1$ while formula (2.13) to (2.16) for $k = 2$.

Proof of Proposition 2.4. It goes by induction. Corollary 2.1 allows to start it. Supposing then the validity of (2.16) for k for all multi-Lie algebras, we observe that

$$C(x_1, \ldots, x_k, u | y_1, \ldots, y_k, u) = 0 \tag{2.17}$$

for any $x_1, \ldots, x_k, y_1, \ldots, y_k, u \in \mathcal{V}$. In fact, this condition coincides with the kth order compatibility condition for $(n-1)$-Lie algebra P_u. In particular,

$$C(v_1, \ldots, v_k, v_{k+1} + w_{k+1} | w_1, \ldots, w_k, v_{k+1} + w_{k+1}) = 0. \tag{2.18}$$

On the other hand, it is easily seen that

$$\begin{aligned} &C(v_1, \ldots, v_k, v_{k+1} + w_{k+1} | w_1, \ldots, w_k, v_{k+1} + w_{k+1}) \\ &\quad = \sum_{I, i_1 = 1} \langle (v, w)_I, v_{k+1} + w_{k+1} | (w, v)_I, v_{k+1} + w_{k+1} \rangle, \end{aligned}$$

where $(v, w)_I$ has the same meaning as in (2.15) and $((v, w)_I, x)$ denotes the sequence that becomes $(v, w)_I$ once the last term x is deleted. Multi-linearity of the symbol $\langle \ldots | \ldots \rangle$ allows to develop last expression as the sum of terms of the form $\langle (v, w)_I, x | (w, v)_I, y \rangle$ with x, y taking independently the values v_{k+1}, w_{k+1} . After that it remains to observe that the kth order compatibility condition for the algebra P_x gives

$$\sum_{I, i_1 = 1} \langle (v, w)_I, x | (w, v)_I, x \rangle = 0 \tag{2.19}$$

and

$$\begin{aligned} &C(v_1, \ldots, v_{k+1} | w_1, \ldots, w_{k+1}) \\ &\quad = \sum_{I, i_1 = 1} \langle (v, w)_I, v_{k+1} | (w, v)_I, w_{k+1} \rangle \\ &\qquad + \sum_{I, i_1 = 1} \langle (v, w)_I, w_{k+1} | (w, v)_I, v_{k+1} \rangle. \qquad \square \end{aligned}$$

Example 2.6. The explicit form of the third compatibility condition is

$$\begin{aligned} &Comp(P_{v_1, v_2, v_3}, P_{w_1, w_2, w_3}) + Comp(P_{v_1, v_2, w_3}, P_{w_1, w_2, v_3}) \\ &\quad + Comp\,(P_{v_1, w_2, v_3}, P_{w_1, v_2, w_3}) + Comp(P_{v_1, w_2, w_3}, P_{w_1, v_2, v_3}) = 0. \end{aligned}$$

The second order compatibility conditions provide some necessary conditions for the following natural question:

Whether two given n-Lie algebra structures Q and R come from a common $(n+1)$-Lie algebra structure, i.e. whether $Q = P_u$, $R = P_v$ for an $(n+1)$-Lie algebra P and some $u, v \in \mathcal{V}$?

Corollary 2.2. *If n-Lie algebra structures Q and R are first order hereditary for an $(n+1)$-Lie algebra, then*

$$Comp(Q_w, R_z) + Comp(Q_z, R_w) = 0 \quad \forall w, z \in \mathcal{V}. \tag{2.20}$$

3. n-Poisson manifolds

The concept of n-Poisson manifold generalizes that of Poisson one ($n = 2$) just in the same sense as n-Lie algebras do with respect to Lie algebras. It was introduced by Takhtajan in [23]. Filippov [7] in his pioneering work gives an example (see Example 3.1) which turns out to be locally equivalent to the general concept in virtue of an analog of the Darboux lemma for n-Poisson structures. This analog was found recently by Alekseevsky and Guha [1]. Below we present a simple purely algebraic proof of it which is valid in more general algebraic contexts, for instance, for smooth algebras. Since n-Poisson structures are special kind of n-Lie algebra ones we can use freely results of Section 2 in this context.

Definition 3.1. Let M be a smooth manifold. An n-Lie algebra structure on $C^\infty(M)$

$$(f_1, \ldots, f_n) \to \{f_1, \ldots, f_n\} \in C^\infty(M), \quad f_i \in C^\infty(M) \tag{3.1}$$

is called an n-Poisson structure on M if the map

$$f \to \{f, \ldots, \} \tag{3.2}$$

is a *derivation* of the algebra $C^\infty(M)$.

Last condition means Leibniz's rule with respect to the first argument

$$\{fg, h_1, \ldots, h_{n-1}\} = f\{g, h_1, \ldots, h_{n-1}\} + g\{f, h_1, \ldots, h_{n-1}\}. \tag{3.3}$$

Evidently, due to skew-symmetry, Leibniz's rule is valid for all arguments.

An equivalent way to express this property is to say that the operator

$$X_{f_1,\ldots,f_{n-1}} : C^\infty(M) \to C^\infty(M) \tag{3.4}$$

defined as

$$X_{f_1,\ldots,f_{n-1}}(g) = \{f_1, \ldots, f_{n-1}, g\} \tag{3.5}$$

is a vector field on M. Such a field is called *Hamiltonian* corresponding to the *Hamiltonian functions* $f_1, \ldots, f_{n-1}$.

A manifold supplied with an n-Poisson structure is called *n-Poisson or Nambu–Poisson manifold.* It is natural to interpret a vector field on M as a 1-Poisson structure on it.

Vector fields on M that are derivations of the considered n-Poisson structure are called *canonical* (with respect to it). As in the classical case $n = 2$ Hamiltonian fields of an n-Poisson structure are, obviously, canonical fields.

Let M and N be n-Poisson manifolds and $\{\,,\,\}_M$ and $\{\,,\,\}_N$ be the corresponding brackets. A map $F : M \to N$ is said to be *Poisson* if

$$\{F^*(f_1), \ldots, F^*(f_n)\}_M = F^*(\{f_1, \ldots, f_n\}_N) \quad \forall f_1, \ldots, f_n \in C^\infty(N) \tag{3.6}$$

Example 3.1 [7]. Let $X_1, \ldots, X_n$ be commuting vector fields on M. Then

$$\{f_1, \ldots, f_n\} = \det \|X_i(f_j)\| \tag{3.7}$$

is an n-Poisson structure on M. More generally, if $\mathcal{A}$ is a commutative algebra any set of n commuting derivations of it defines an n-Poisson structure on it. Note also that the so-defined n-Poisson structure is invariant with respect to a unimodular transformation of fields $Y_i = \sum_j s_{ij} X_j$, $\det \|s_{ij}\| = 1$, $s_{ij} \in C^\infty(M)$.

More generally, if $[X_j,\ X_k] = c^l_{jk} X_l$, $\quad c^l_{jk} \in C^\infty(M)$, then we have $\{f_1, \ldots, f_n\} = \det \|X_i(f_j)\|$ is an n-Poison structure on M.

n-Poisson structures are multi-derivations, i.e. multi-linear operators on the algebra $C^\infty(M)$ which are derivations with respect to any of their arguments. This is a particular case of the general concept of multi-differential operator on $C^\infty(M)$ (more generally, on a commutative algebra $\mathcal{A}$ [25]). It means that for any $i = 1, 2, \ldots, k$ the correspondence

$$f \to \Delta(f_1, \ldots, f_{i-1}, f, f_{i+1}, \ldots, f_k) \tag{3.8}$$

is a differential operator for any fixed set of functions $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k$. When dealing with multi-differential operators and, in particular, with multi-derivations we will adopt the notation of Section 2. For instance, we write $f \rfloor$ or ι_f for the insertion operator, so that if Δ is a k-differential operator, then $f \rfloor \Delta = \iota_f(\Delta) = \Delta_f$ are three different notations for the $(k-1)$-differential operator

$$(f \rfloor \Delta)(g_1, \ldots, g_{k-1}) = \Delta(f, g_1, \ldots, g_{k-1}). \tag{3.9}$$

Note the one-to-one correspondence between k-contravariant tensors T and k-derivations Δ given as

$$\mathrm{d}f_k \rfloor \cdots \rfloor \mathrm{d}f_1 \rfloor T = T(\mathrm{d}f_1, \ldots, \mathrm{d}f_k) = \Delta(f_1, \ldots, f_k). \tag{3.10}$$

If, moreover, T is skew-symmetric, then it is a k-vector. In particular, an n-Poisson structure can be given either by a skew-symmetric n-derivation, or by the k-vector corresponding to it .

The mentioned one-to-one correspondence between skew-symmetric multi-derivations and multi-vectors allows to carry well-known operations from the latters over the formers. For instance, the standard *wedge product* of two multi-vectors allows to define *the wedge product* of the corresponding multi-derivations Δ and ∇ as

$$(\Delta \wedge \nabla)(f_1, \ldots, f_{k+l}) = \sum_I (-1)^{(I,\bar{I})} \Delta(f_I) \nabla(f_{\bar{I}}), \tag{3.11}$$

where $I = (i_1, \ldots, i_k)$, $1 \le i_1 \le \cdots \le i_k \le k + l$, is an increasing subsequence of integers, $\bar{I}$ is its complement in $\{1, 2, \ldots, k+l\}$, $(I, \bar{I})$ is the corresponding permutation of

$1, 2, \dots, k+l$, $(-1)^{(I,\bar{I})}$ stands for the sign of it and f_I (respectively $f_{\bar{I}}$) is a shortnoting for $f_{i_1}, \dots, f_{i_k}$ (respectively $f_{\bar{i}_1}, \dots, f_{\bar{i}_l}$). Moreover, definition (3.11) makes sense, in fact, for arbitrary multi-differential operators, not necessarily derivation, and therefore, defines an associative and graded commutative multiplication over them.

The Schouten–Nijenhuis bracket carried over multi-derivations looks as

$$\begin{aligned}\lceil \Delta, \nabla \rfloor (f_1, \dots, f_{k+l-1}) = &\sum_{|I|=k-1} (-1)^{(I,\bar{I})} \Delta(f_I, \nabla(f_{\bar{I}})) \\ &- \sum_{|J|=k} (-1)^{(J,\bar{J})} \nabla(\Delta(f_J), f_{\bar{J}}),\end{aligned} \tag{3.12}$$

where I and J stand, as before, for increasing subsequences of $\{1, 2, \dots, k+l-1\}$ while $|I|$ (respectively, $|J|$) denotes the length of I (respectively, J). Similarly to (3.11), formula (3.12) remains meaningful for arbitrary skew-symmetric multi-differential operators and this way the Schouten–Nijenhuis bracket is extended on them. More exactly, defining the *Schouten grading* of k-differential operators to be equal to $k-1$, we have:

Proposition 3.1. *The Schouten graded skew-symmetric multi-differential operators supplied with the bracket operation* (3.12) *form a graded Lie algebra, i.e.*

$$\lceil \Delta, \nabla \rfloor = -(-1)^{(k-1)(l-1)} \lceil \nabla, \Delta \rfloor \tag{3.13}$$

(*graded skew-symmetry*) *and*

$$\begin{aligned}&(-1)^{(k-1)(m-1)} \lceil \Delta, \lceil \nabla, \Box \rfloor \rfloor + (-1)^{(m-1)(l-1)} \lceil \Box, \lceil \Delta, \nabla \rfloor \rfloor \\ &\quad + (-1)^{(l-1)(k-1)} \lceil \nabla, \lceil \Box, \Delta \rfloor \rfloor = 0\end{aligned}$$

(*graded Jacobi identity*).

Proof. Graded skew-commutativity is obvious while the graded Jacobi identity is checked by a direct but tedious computation. □

Corollary 3.1. *The well-known compatibility condition* $\lceil \Delta, \nabla \rfloor = O$ *of two Poisson structures* $\Delta(f, g) = \{f, g\}_{\mathrm{I}}$ *and* $\nabla(f, g) = \{f, g\}_{\mathrm{II}}$ *is in the considered context identical to the one given in the preceding section.*

Proof. Just to compare (2.10) for $n = 2$ and (3.12) for $k = l = 2$. □

Remark 3.1. It is worth to emphasize that the Lie derivative of a multi-vector V corresponds in the aforementioned sense to the Lie derivative of the multi-derivation Δ associated with V in the sense of the previous section. In particular, the fact that V is an n-Poisson multi-vector can be seen as

$$X_{f_1, \dots, f_{n-1}}(V) = 0, \tag{3.14}$$

where $X(V)$ is a short notation for the Lie derivative $L_X(V)$ of V we shall use to simplify some formulae. Similarly, the compatibility condition of two n-vectors V and W can be written in the form

$$Y_{f_1,\dots,f_{n-1}}(V) + X_{f_1,\dots,f_{n-1}}(W) = 0 \tag{3.15}$$

where $X_{f_1,\dots,f_{n-1}}$ and $Y_{f_1,\dots,f_{n-1}}$ are Hamiltonian vector fields with the same *Hamilton functions* $f_1, \dots, f_{n-1}$ with respect to Poisson structures given by V and W, respectively.

A function $g \in C^\infty(M)$ is said to be a *Casimir function* if

$$X_{f_1,\dots,f_{n-1}}(g) = \{f_1, \dots, f_{n-1}, g\} = 0, \quad \forall f_1, \dots, f_{n-1} \in C^\infty(M). \tag{3.16}$$

All Casimir functions form, evidently, a subalgebra $\mathcal{K}$ of $C^\infty(M)$. We denote it also by $Cas(P)$ when it becomes necessary to refer to the n-Poisson structure P in question and call it the *Casimir algebra*. An ideal $\mathcal{I}$ of the Casimir algebra allows to restrict the original n-Poisson structure to the submanifold (possibly with singularities)

$$N = \{x \in M \mid f(x) = 0, f \in \mathcal{I}\} \subseteq M. \tag{3.17}$$

To see this note that

$$C^\infty(N) = C^\infty(M)/\mathcal{I}C^\infty(M) \tag{3.18}$$

if N is a submanifold without singularities. Otherwise, define the smooth function algebra on N by means of (3.18). Further note that the ideal $\mathcal{I}^* = \mathcal{I}C^\infty(M) \subseteq C^\infty(M)$ is *stable* (with respect to the n-Poisson structure in question) in the sense that $\{f_1, \dots, f_{n-1}, g\} \in \mathcal{I}^*$ if $g \in \mathcal{I}^*$. This allows one to define the restricted n-Poisson structure on N just by passing to quotients

$$\{\tilde{f}_1, \dots, \tilde{f}_n\}_N = \widetilde{\{f_1, \dots, f_n\}}, \tag{3.19}$$

where $\tilde{f}_i = f_i (\text{mod } \mathcal{I}^*)$. From a geometrical point of view the stability of $\mathcal{I}^*$ implies that Hamiltonian vector fields are tangent to N. The smallest such submanifolds N correspond to the largest, i.e. maximal, ideals of $\mathcal{K}$. Since any *non-wild* maximal ideal of $\mathcal{K}$ is of the form $\mathcal{I} = \ker G$ where $G : \mathcal{K} \to \mathbb{R}$ is a $\mathbb{R}$-homomorphism of unitary $\mathbb{R}$-algebras it is reasonable to limit our considerations to these ones. Denote by N_G the submanifold of M corresponding to the ideal $\mathcal{I} = \ker G$ and recall that all $\mathbb{R}$-homomorphisms of $\mathcal{K}$ constitute a manifold (with singularities) $Spec_{\mathbb{R}}\mathcal{K}$, the real spectrum of $\mathcal{K}$, in such a way that $\mathcal{K} = C^\infty(Spec_{\mathbb{R}}\mathcal{K})$. We shall call it the *Casimir manifold* of the considered n-Poisson structure and denote it by $Cas(M)$ or $Cas(P)$ depending on the context. Then the canonical embedding $\mathcal{K} \subseteq C^\infty(M)$ induces by duality the Casimir map

$$Cas : M \to Cas(M). \tag{3.20}$$

By construction $N_G = Cas^{-1}(G)$. This way one gets the *Casimir fibration* of M whose fibers are n-Poisson manifolds. In the Casimir fibration is canonically inscribed the *Hamiltonian foliation* which is defined as follows. First, note that the commutator of two Hamiltonian fields is a sum of Hamiltonian fields. In fact, formula (2.7) in the considered context looks as

$$[X_{f_1,\dots,f_{n-1}}, X_{g_1,\dots,g_{n-1}}] = \sum_i X_{g_1,\dots,\{f_1,\dots,f_{n-1},g_i\},\dots,g_{n-1}}. \tag{3.21}$$

This implies that the $C^\infty(M)$-module $H(P)$ of vector fields generated by all Hamiltonian ones is closed with respect to the Lie commutator operation. It defines, therefore, a (singular) foliation on M called Hamiltonian. It was already mentioned that Hamiltonian fields are tangent to submanifolds N_G . Hence, any *Hamiltonian leaf*, i.e. that of the Hamiltonian foliation, belongs to a suitable Casimir submanifold N_G. So, Casimir submanifolds are foliated by *Hamiltonian leaves*.

Example 3.2. Let T^{n+1} be the standard $(n+1)$-dimensional torus with standard angular coordinates $\theta_1, \theta_2, \dots, \theta_{n+1}$. Consider the n-Poisson structure on it defined by vector fields

$$X_1 = \frac{\partial}{\partial\theta_1} + \lambda\frac{\partial}{\partial\theta_2}, X_2 = \frac{\partial}{\partial\theta_3}, \dots, X_n = \frac{\partial}{\partial\theta_{n+1}} \tag{3.22}$$

as in Example 3.1. Then for a rational λ $Cas(T^{n+1}) = S^1$ and the Casimir map $Cas : T^{n+1} \to S^1$ is a trivial fiber bundle with T^n as a fibre. In this case fibres of the Casimir map are identical to leaves of the Hamiltonian foliation. If λ is irrational, then $Cas(T^{n+1})$ is just a point which is equivalent to $\mathcal{K} = \mathbb{R}$. In other words, T^{n+1} is the unique submanifold of the form N_G. On the other hand, the Hamiltonian foliation in this case is n-dimensional and its leaves are copies of $\mathbb{R}^n$ immersed everywhere densely in T^{n+1}.

Since Hamiltonian vector fields are, by construction, tangent to the leaves of the Hamiltonian foliation, the Poisson multi-vector of the considered Poisson structure is also tangent to them. For this reason on any such leaf there exists a unique n-Poisson structure such that the canonical immersion $L \hookrightarrow M$ becomes an n-Poisson map. In Section 4 it will be shown that Poisson leaves are either n-dimensional (regular), or 0-dimensional (singular) if $n > 2$ what is in strong contrast with the classical case $n = 2$. By this reason n-Poisson structures on n-dimensional manifolds are to be described. We will get it as a particular case of the following general assertion.

Proposition 3.2. *Let P be an n-Poisson structure of rank n on a manifold M. Then for any $f \in C^\infty(M)$, fP is an n-Poisson structure and any two structures of this form are compatible.*

Proof. It is based on the general formula

$$L_{fX}(Q) = fL_X(Q) - X \wedge (f \rfloor Q) \tag{3.23}$$

for any $f \in C^\infty(M)$, $X \in \mathcal{D}(M)$ and a multi-vector Q on M (see, for instance, [2]). By applying it to $X = P_{h_1,\dots,h_{n-1}}$ and $Q = gP, g \in C^\infty(M)$, and taking into account that $P_{h_1,\dots,h_{n-1}}(P) = 0$ one finds

$$(fP)_{h_1,\dots,h_{n-1}}(gP) = fP_{h_1,\dots,h_{n-1}}(g)P - P_{h_1,\dots,h_{n-1}} \wedge (gP_f). \tag{3.24}$$

This formula allows to rewrite the compatibility condition (2.10) for fP and gP as

$$\begin{aligned}(fP)_{h_1,\dots,h_{n-1}}(gP) + (gP)_{h_1,\dots,h_{n-1}}(fP) \\ = fP_{h_1,\dots,h_{n-1}}(g)P + gP_{h_1,\dots,h_{n-1}}(f)P - P_{h_1,\dots,h_{n-1}} \wedge (gP_f + fP_g) \\ = P_{h_1,\dots,h_{n-1}}(fg)P - P_{h_1,\dots,h_{n-1}} \wedge ((fg)\rfloor P) = (fg)\rfloor (P_{h_1,\dots,h_{n-1}} \wedge P),\end{aligned}$$

It remains to note that $P_{h_1,\dots,h_{n-1}} \wedge P = 0$ for a multi-vector of rank n. □

Corollary 3.2. *Any Frobenius n-vector field V on a manifold M is an n-Poisson one. In particular, such are n-vector fields on an n-dimensional manifold M.*

Proof. Since V defines an n-dimensional distribution (with singularities) on M it can be locally presented as $V = hX_1 \wedge \cdots \wedge X_n$ for a suitable $h \in C^\infty(M)$. But $X_1 \wedge \cdots \wedge X_n$ is just the Poisson structure of Example 3.1 and, so, V is also an n-Poisson structure in virtue of Proposition 3.2. □

4. Decomposability of n-Poisson structures

In this section we prove a result which, in a sense, is an analog of the Darboux lemma for n-Poison structures with $n > 2$. It tells that the rank of a non-trivial Poisson n-vector is equal to n and, therefore, such an n-vector is locally decomposable. This was conjectured by Takhtajan and proved recently by Alexeevsky and Guha [1]. Our approach is, however, quite different. We start with collecting and recalling some elementary facts of multi-linear algebra.

Let $\mathcal{V}$ be a finite-dimensional vector space. Denote by $\Lambda^k(\mathcal{V})$ its kth exterior power and put

$$V_{a_1,\dots,a_l} := a_l \rfloor \cdots \rfloor a_1 \rfloor V \in \Lambda^{k-l}(\mathcal{V}) \tag{4.1}$$

for $V \in \Lambda^k(\mathcal{V})$ and $a_1, \dots, a_l \in \mathcal{V}^*$.

The following is well-known.

Lemma 4.1. *A non-zero k-vector $V \in \Lambda^k(\mathcal{V})$ is decomposable, i.e. $V = v_1 \wedge \cdots \wedge v_k$, for some $v_i \in \mathcal{V}$, iff it is of rank k.*

Vectors v_i's are defined uniquely up to a unimodular transformation $v_i \to w_i = \sum_j c_{ij} v_j$. The subspace of $\mathcal{V}$ generated by $v_1, \dots, v_k$ coincides with that generated by all vectors of the form $V_{a_1,\dots,a_{k-1}} \in \mathcal{V}$.

Recall also the following lemma.

Lemma 4.2. *If $v \wedge V = 0$, $v \in \mathcal{V}$, $V \in \Lambda^k(\mathcal{V})$, then V is factorized by v, i.e. $V = v \wedge V'$ for a $V' \in \Lambda^{k-1}(\mathcal{V})$.*

Together with Lemma 4.1 this implies the following.

Lemma 4.3. *A k-vector V is decomposable iff*

$$V_{a_1,\dots,a_{k-1}} \wedge V = 0, \quad \forall a_1, \dots, a_{k-1} \in \mathcal{V}^*.$$

Lemma 4.4 (on three planes). *Let Π_1, Π_2, Π_3 be $(k-1)$-dimensional subspaces of $\mathcal{V}$ such that* $\dim(\Pi_i \cap \Pi_j) = k-2$ *for $i \neq j$. If $k > 2$, then*
- *the span Π of Π_1, Π_2, Π_3 is k-dimensional,*
- *any $(k-1)$-dimensional subspace Π' of $\mathcal{V}$ intersecting each of Π_i's along a not less than $(k-2)$-dimensional subspace belongs to Π.*

Proof. Obvious. □

Proposition 4.1. *Let V be a k-vector, $k > 2$. If*

$$V_{a,c_1,\dots,c_{k-2}} \wedge V_b + V_{b,c_1,\dots,c_{k-2}} \wedge V_a = 0, \quad \forall a, b, c_1, \dots, c_{k-2} \in \mathcal{V}^*, \tag{4.2}$$

then V is decomposable.

Proof. By putting $a = b$ in (4.2) we see that $W_{c_1,\dots,c_{k-2}} \wedge W = 0$ for $W = V_a$. Therefore, according to Lemma 4.3, the $(k-1)$-vector V_a is decomposable $\forall a \in \mathcal{V}^*$.

Denote now by Π_a the $(k-1)$-dimensional subspace of $\mathcal{V}$ canonically associated, according to Lemma 4.1, with the decomposable $(k-1)$-vector V_a assumed to be different from zero. If $V_{a,c_1,\dots,c_{k-2}} \wedge V_b = 0$ for all $c_1, \dots, c_{k-2} \in \mathcal{V}^*$, then $\Pi_a = \Pi_b$ as it results from Lemmas 4.2 and 4.1. If, otherwise, $V_{a,c_1,\dots,c_{k-2}} \wedge V_b \neq 0$ consider the subspace Π associated according to Lemma 4.1 with the decomposable k-vector $V_{a,c_1,\dots,c_{k-2}} \wedge V_b$. Obviously, $\Pi \supset \Pi_b$.

On the other hand, equality (4.2) shows that Π coincides with the subspace associated with the decomposable k-vector $V_{b,c_1,\dots,c_{k-2}} \wedge V_a = 0$. By this reason $\Pi \supset \Pi_a$ and, therefore, $\dim(\Pi_a \cap \Pi_b) \geq k-2 > 0$. Moreover, if $V_{a,b} \neq 0$, then $\dim(\Pi_a \cap \Pi_b) = k-2$. In fact, $\dim(\Pi_a \cap \Pi_b) = k-1$ implies that $\Pi_a = \Pi_b$ and, hence, $V_a = \lambda V_b$. Hence, $V_{a,b} = \lambda V_{b,b} = 0$ which is impossible.

Observe, finally, that since $V \neq 0$ and $k \geq 3$ there exist $a, b, c \in \mathcal{V}^*$ such that $V_{a,b,c} \neq 0$. In such a situation $(k-2)$-vectors $V_{a,b}$, $V_{b,c}$ and $V_{a,c}$ are different from zero. Hence, as we have already seen previously, mutual intersections Π_a, Π_b and Π_c are all $(k-2)$-dimensional. So, these three subspaces satisfy the hypothesis of Lemma 4.4. By this reason the span Π of them contains all subspaces Π_d, $d \in \mathcal{V}^*$, and consequently all derived vectors $V_{d,d_1,\dots,d_{k-2}}$ belong to Π. Now Lemma 4.1 implies the desired result. □

Our next task is to show that the hypothesis of Proposition 4.1 is satisfied by any n-Poisson multi-vector. First, we need the following property of Lie derivations.

Lemma 4.5. *Let $X \in \mathcal{D}(M)$ and $f \in C^\infty(M)$. For a multi-derivation Δ it holds*

$$L_{fX}(\Delta) = f L_X(\Delta) - X \wedge \Delta_f. \tag{4.3}$$

Proof. By the definition of the Lie derivative we have

$$
\begin{aligned}
L_{fX}(\Delta)(g_1,\dots,g_n) &= fX(\Delta(g_1,\dots,g_n)) - \sum_i \Delta(g_1,\dots,fX(g_i),\dots,g_n)\\
&= f(X(\Delta(g_1,\dots,g_n) - \sum_i \Delta(g_1,\dots,X(g_i),\dots g_n))\\
&\quad - \sum_i (-1)^{i-1} X(g_i)\Delta(f,g_1,\dots,g_n).
\end{aligned}
$$

It remains to note that last sum is just the product $X \wedge \Delta_f$ evaluated on $g_1,\dots,g_n$. □

Next identity is basic.

Proposition 4.2. *Let Δ be an n-derivation. Then for any $f, g, \phi_i \in C^\infty(M)$ it holds:*

$$
\begin{aligned}
\Delta_{fg,\phi_1,\dots,\phi_{n-2}}(\square) = {}& f\Delta_{g,\phi_1,\dots,\phi_{n-2}}(\square)\\
&+ g\Delta_{f,\phi_1,\dots,\phi_{n-2}}(\square) - \Delta_{f,\phi_1,\dots,\phi_{n-2}} \wedge \square_g\\
&- \Delta_{g,\phi_1,\dots,\phi_{n-2}} \wedge \square_f.
\end{aligned}
\tag{4.4}
$$

Proof. First, note that $\Delta_{fg} = f\Delta_g + g\Delta_f$. So one has

$$\Delta_{fg,\phi_1,\dots,\phi_{n-2}}(\square) = (f\Delta_{g,\phi_1,\dots,\phi_{n-2}})(\square) + (g\Delta_{f,\phi_1,\dots,\phi_{n-2}})(\square).$$

On the other hand, by putting $Y = \Delta_{f,\phi_1,\dots,\phi_{n-2}}$, $Z = \Delta_{g,\phi_1,\dots,\phi_{n-2}}$ and applying Lemma 4.5 one finds

$$
\begin{aligned}
\Delta_{fg,\phi_1,\dots,\phi_{n-2}}(\Delta_f) &= (L_{gY} + L_{fZ})(\square)\\
&= gL_Y(\square) + fL_Z(\square) - Y \wedge \square_g - Z \wedge \square_f. \qquad \square
\end{aligned}
\tag{4.5}
$$

Corollary 4.1. *If Δ is an n-Poisson structure, then for any $f, g, \phi_i \in C^\infty(M)$ it holds*

$$\Delta_{f,\phi_1,\dots,\phi_{n-2}} \wedge \Delta_g + \Delta_{g,\phi_1,\dots,\phi_{n-2}} \wedge \Delta_f = 0. \tag{4.6}$$

Proof. Formula (4.4) for an n-Poisson Δ, and $\square = \Delta$ is reduced, obviously, to (4.6). □

Remark 4.1. Formula (4.6) for $n = 2$ becomes empty. We mention also the following particular case of (4.6) for which $g = f$:

$$\Delta_{g,\phi_1,\dots,\phi_{n-2}} \wedge \Delta_g = 0. \tag{4.7}$$

Theorem 4.1. *Any non-trivial n-Poisson n-vector V is of rank n if n > 2.*

Proof. Formula (4.6) can be rewritten as

$$
\begin{aligned}
&(\mathrm{d}\phi_{n-1} \rfloor \cdots \rfloor \mathrm{d}\phi_1 \rfloor \mathrm{d}f \rfloor V) \wedge (\mathrm{d}g \rfloor V)\\
&\quad + (\mathrm{d}\phi_{n-1} \rfloor \cdots \rfloor \mathrm{d}\phi_1 \rfloor \mathrm{d}g \rfloor V) \wedge (\mathrm{d}f \rfloor V) = 0.
\end{aligned}
$$

Evaluated at a point $x \in M$ it ensures the hypothesis (4.2) of Proposition 4.1 for the n-vector V_x over the tangent space $\mathcal{V} = T_x M$. Therefore, V_x is of rank n or otherwise identically equal to zero. □

Corollary 4.2. *For $n > 2$ regular leaves of the Hamiltonian foliation of an n-Poisson manifold are n-dimensional. Its singular leaves are just points.*

Remark 4.2. Since an n-dimensional foliation can be given by means of n commuting vector fields in a neighborhood of its regular point, Example 3.1 exhausts regular local forms of n-Poisson structures for $n > 2$.

Another eventually very important consequence of Theorem 4.1 is that the cartesian product of two n-Poisson manifolds is not in a natural way such a one if $n > 2$. In fact, there is no natural way to construct an n-dimensional foliation on the cartesian product of two manifolds supplied with such ones.

Theorem 4.1 shows n-Poisson structures for $n > 2$ to be extremely rigid what implies some peculiarities going beyond the *binary* based expectations. Below we exhibit two of them: no cartesian products and no (in general) n-Poisson structure on the dual of an n-Lie algebra.

First, note that given two n-vector fields P and Q on manifolds M and N, respectively, their direct sum $P \oplus Q$ which is an n-vector field on $M \times N$ is naturally defined.

Corollary 4.3. *If P and Q are non-trivial n-Poisson vector fields, then $P \oplus Q$ is not an n-Poisson one for $n > 2$.*

Proof. Just to note that $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$. □

This result can be also proved by a direct computation.

Second, given an n-Lie algebra structure $[\cdot, \ldots, \cdot]$ on $\mathcal{V}$ one can try to associate with it an n-Poisson structure on its dual $\mathcal{V}^*$ just by copying the standard construction for $n = 2$. Namely, let $x_1, \ldots, x_N \in \mathcal{V}$ be a basis. Interpreting x_i's to be coordinate functions on $\mathcal{V}^*$, let us put

$$T = \sum_{1 \le i_1 < \cdots < i_n \le N} [x_{i_1}, \ldots, x_{i_n}] \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_n}}. \tag{4.8}$$

In a coordinate-free form the n-vector field T can be presented as

$$T(\mathrm{d}f_1, \ldots, \mathrm{d}f_n)(u) = [\mathrm{d}_u f_1, \ldots, \mathrm{d}_u f_n]$$

with $u \in \mathcal{V}^*$ and $f_i \in C^\infty(\mathcal{V}^*)$ where the differential $\mathrm{d}_u f_i$ of f_i at the point u is interpreted canonically to be an element of $\mathcal{V}$. This n-vector field T is called *associated* with the n-Lie algebra structure in question.

It is well known (for instance, [26]) that formula (4.8) defines the standard Poisson structure on $\mathcal{V}^*$ when $n = 2$. However, it is no longer so when $n > 2$.

Corollary 4.4. *If $n > 2$, the n-vector field T given by* (4.8) *is not generally an n-Poisson one.*

Proof. First, note that the n-vector field associated with the direct product of two n-Lie algebras is the direct sum of n-vector fields associated with each of them. Since, obviously, the n-vector field associated with a non-trivial n-Lie algebra is of rank not less than n, the n-vector field associated with the product of two non-trivial n-Lie algebras is of rank not less than $2n$. Therefore, it cannot be an n-Poisson vector if $n > 2$. □

On the other hand we have:

Proposition 4.3. *Formula* (4.8) *defines an n-Poisson structure on the dual of an n-Lie algebra of dimension $\leq n+1$.*

Proof. As it is easy to see any n-vector defined on a space of dimension $\leq n+1$ is either of rank n or 0; so, under the hypothesis of the proposition T defines an n- or 0-dimensional distribution on $\mathcal{V}^*$. Denote by Δ the n-derivation on $\mathcal{V}^*$ corresponding to T as in (4.8). It suffices to show that

$$\Delta_{f_1,\dots,f_{n-1}}(\Delta) = 0 \tag{4.9}$$

for any system of polynomials $f_i(x)$ in variables x_k's. We prove it by induction on the total degree $\delta = \deg f_1 + \cdots + \deg f_{n-1}$ by starting from $\delta = n-1$.

To start the induction note that in the case all f_i's are linear on $\mathcal{V}^*$, i.e. elements of $\mathcal{V}$, identity (4.9) is identical to the n-Jacobi identity of the original n-Lie algebra.

To complete the induction it is sufficient to show that (4.9) holds for the system $f_1 = gh, f_2, \dots, f_{n-1}$ if it holds for $g, f_2, \dots, f_{n-1}$ and $h, f_2, \dots, f_{n-1}$. Taking into account that $\Delta_{gh,f_2,\dots,f_{n-1}} = g\Delta_{h,f_2,\dots,f_{n-1}} + h\Delta_{g,f_2,\dots,f_{n-1}}$ and Lemma 4.5 the problem is reduced to prove that

$$\Delta_{g,f_2,\dots,f_{n-1}} \wedge \Delta_h + \Delta_{h,f_2,\dots,f_{n-1}} \wedge \Delta_g = 0. \tag{4.10}$$

But since T is of rank n $\Delta_{\varphi_1,\dots,\varphi_{n-1}} \wedge \Delta = 0$ for any system $\varphi_1, \dots, \varphi_{n-1} \in C^\infty(\mathcal{V}^*)$. So we have

$$0 = h \rfloor (\Delta_{g,f_2,\dots,f_{n-1}} \wedge \Delta) = \Delta(g, f_2, \dots, f_{n-1}, h)\Delta - \Delta_{g,f_2,\dots,f_{n-1}} \wedge \Delta_h,$$

so that

$$\Delta_{g,f_2,\dots,f_{n-1}} \wedge \Delta_h = \Delta(g, f_2, \dots, f_{n-1}, h),$$

and, similarly,

$$\Delta_{h,f_2,\dots,f_{n-1}} \wedge \Delta_g = \Delta(h, f_2, \dots, f_{n-1}, g).$$

Hence, (4.10) results from skew-symmetry of Δ. □

Previous discussions lead us to conjecture that:

If $n > 2$ any n-Lie algebra is split into the direct product of a trivial n-Lie algebra and a number of non-trivial n-Lie algebras of dimensions n and $(n+1)$.

In fact, this conjecture saves in essence the fact that an n-Lie algebra structure generates an n-Poisson structure on its dual (Proposition 4.3) in view of the resistance of n-Poisson manifolds to form cartesian products (Corollary 4.3) if $n > 2$. Also, at least to our knowledge, all known examples in the literature are in favor of this conjecture.

Finally, mention an alternative (and also natural) way to save the dual construction by giving to the concept of n-Poisson manifold the *dual* meaning (see [27]). For fundamentals of this dual approach we send the reader to [19]. A discussion of the Koszul duality can be found in [9,15].

We conclude this section by answering the natural question: what are *multiplicative* compatibility conditions for two multi-Poisson structures, i.e. conditions ensuring that their wedge product is again a multi-Poisson one.

Proposition 4.4. *Let Δ and ∇ be multi-Poisson structures on the manifold M whose multiplicities coincide with their ranks (for instance, they are of multiplicities greater than 2). Then $\Delta \wedge \nabla$ is a multi-Poisson structure on M iff $\lceil \Delta, \nabla \rfloor = 0$, $\Delta_{g_1,\dots,g_{k-1}}(\nabla) \wedge \nabla = 0$, $\nabla_{h_1,\dots,h_{l-1}}(\Delta) \wedge \Delta = 0$ for all $g_i, h_j \in C^\infty(M)$, k and l denoting the multiplicities of Δ and ∇, respectively.*

Proof. First, note the formula which is a direct consequence of the wedge product definition:

$$(\Delta \wedge \nabla)_{f_1,\dots,f_N} = \sum_I (-1)^{(k-|I|)(N-|I|)+(I,\bar{I})} \Delta_{f_I} \wedge \nabla_{f_{\bar{I}}}, \tag{4.11}$$

where I runs all ordered subsets of $\{1,\dots,N\}$ and $|I|$ denotes the cardinality of I. In particular, for $N = k + l - 1$ we have

$$(\Delta \wedge \nabla)_{f_1,\dots,f_{k+l-1}} = \sum_{|I|=k} (-1)^{(I,\bar{I})} \Delta(f_I) \nabla_{f_{\bar{I}}} + \sum_{|I|=k-1} (-1)^{l+(I,\bar{I})} \nabla(f_{\bar{I}}) \Delta_{f_I}. \tag{4.12}$$

By applying Lemma 4.5 to $f = \Delta(f_I)$, $X = \nabla_{f_{\bar{I}}}$ and taking into account that $\nabla_{f_{\bar{I}}}(\nabla) = 0$ and $\nabla_{f_{\bar{I}}} \wedge \nabla = 0$ (∇ is l-Poisson of rank l) we find

$$(\Delta(f_I)\nabla_{f_{\bar{I}}})(\Delta \wedge \nabla) = \Delta(f_I)\nabla_{f_{\bar{I}}}(\Delta) \wedge \nabla - (-1)^k \nabla_{f_{\bar{I}}} \wedge \Delta \wedge \nabla_{\Delta(f_I)} \tag{4.13}$$

and, similarly,

$$(\nabla(f_{\bar{I}})\Delta_{f_I})(\Delta \wedge \nabla) = \nabla(f_{\bar{I}})\Delta \wedge \Delta_{f_I}(\nabla) - \Delta_{f_I} \wedge \Delta_{\nabla(f_{\bar{I}})} \wedge \nabla. \tag{4.14}$$

Since $\Delta_{f_I} \wedge \Delta = 0$, then

$$0 = \nabla(f_{\bar{I}}) \rfloor (\Delta_{f_I} \wedge \Delta) = \Delta(f_I, \nabla(f_{\bar{I}}))\Delta - \Delta_{f_I} \wedge \Delta_{\nabla(f_{\bar{I}})},$$

i.e.

$$\Delta_{f_I} \wedge \Delta_{\nabla(f_{\bar{I}})} = \Delta(f_I, \nabla(f_{\bar{I}})) \tag{4.15}$$

and, similarly,

$$\nabla_{f_{\bar{I}}} \wedge \nabla_{\Delta(f_I)} = \nabla(f_{\bar{I}}, \Delta(f_I)). \tag{4.16}$$

Now bearing in mind (4.12)–(4.16) we get

$$\begin{aligned}
&(\Delta \wedge \nabla)_{f_1,\dots,f_{k+l-1}}(\Delta \wedge \nabla) \\
&= \sum_{|I|=k} (-1)^{(I,\bar{I})}(\Delta(f_I)\nabla_{f_{\bar{I}}}(\Delta) \wedge \nabla - \nabla(f_{\bar{I}}, \Delta(f_I))\Delta \wedge \nabla) \\
&\quad + \sum_{|I|=k-1} (-1)^{l+(I,\bar{I})}(\nabla(f_{\bar{I}})\Delta \wedge \Delta_{f_I}(\nabla) - \Delta(f_I, \nabla(f_{\bar{I}}))\Delta \wedge \nabla) \\
&= \sum_{|I|=k} (-1)^{(I,\bar{I})}\Delta(f_I)\nabla_{f_{\bar{I}}}(\Delta) \wedge \nabla + \sum_{|I|=k-1} (-1)^{l+(I,\bar{I})}\nabla(f_{\bar{I}})\Delta \wedge \Delta_{f_I}(\nabla) \\
&\quad - (-1)^l \lceil \Delta, \nabla \rfloor (f_1, \dots, f_{k+l-1})
\end{aligned} \tag{4.17}$$

(see (3.12)). If $\Delta \wedge \nabla$ is a multi-Poisson structure, then it is also a multi-Poisson structure in the dual sense defined in [19]. But for such structures $\Delta \wedge \nabla$ is multi-Poisson iff $\lceil \Delta, \nabla \rfloor = 0$. This shows that $\lceil \Delta, \nabla \rfloor = 0$ is a necessary condition for the considered problem.

Observe now that due to local decomposability of multi-vectors corresponding to Δ and ∇ the product $\Delta \wedge \nabla$ is different from zero iff they are transversal to each other. This implies that the leaves of the corresponding Hamiltonian foliations intersect one another transversally. By this reason one can find k local Casimir functions of ∇, say $f_1, \dots, f_k$, such that $\Delta(f_1, \dots, f_k) \neq 0$ and l Casimir functions of Δ, say $f_{k+1}, \dots, f_{k+l}$ such that $\nabla(f_{k+1}, \dots, f_{k+l}) \neq 0$. For such chosen f_i's all summands of the first two summations of (4.17) vanish except one which is

$$\Delta(f_1, \dots, f_k)\nabla_{f_{k+1},\dots,f_{k+l-1}}(\Delta) \wedge \nabla.$$

This implies $\nabla_{f_{k+1},\dots,f_{k+l-1}}(\Delta) \wedge \nabla = 0$, if $\Delta \wedge \nabla$ is $(k+l)$-Poisson. Observing then that local Casimir functions of both Δ and ∇ generate in that situation a local smooth function algebra, one can conclude that

$$\nabla_{g_1,\dots,g_{l-1}}(\Delta) \wedge \nabla = 0 \tag{4.18}$$

for any family of functions $g_1, \dots, g_{l-1}$.

Similarly, it is proved that

$$\Delta_{g_1,\dots,g_{l-1}}(\nabla) \wedge \Delta = 0. \tag{4.19}$$

This shows that (4.18), (4.19) and $\lceil \Delta, \nabla \rfloor = 0$ are necessary. The sufficiency is obvious from (4.17). □

5. Local n-Lie algebras

In this section we discuss the most general natural synthesis of the concept of a multi-Lie algebra and that of a smooth manifold which is as follows.

Definition 5.1. A local n-Lie algebra structure on a manifold M is an n-Lie algebra structure

$$(f_1, \ldots, f_n) \to [f_1, \ldots, f_n]$$

on $C^\infty(M)$ which is a multi-differential operator.

Below we continue to use the *operator* notation as well as the *bracket* one for local n-ary structures

$$\Delta(f_1, \ldots, f_n) = [f_1, \ldots, f_n]$$

and refer to the multi-differential operator Δ as the structure in question itself.

Example 5.1. n-Poisson structures are local n-Lie algebra ones.

A well-known result by Kirillov [11] says that for $n = 2$ the bi-differential operator giving a local Lie algebra structure on a manifold M is of first order with respect to both its arguments. An interesting algebraic proof of this fact can be found in [10]. Kirillov's theorem is generalized immediately to higher local multi-Lie algebras.

Proposition 5.1. *Any local n-Lie algebra, $n \geq 2$, is given by an n-differential operator of first order, i.e. of first order with respect to each of its argument.*

Proof. It results from Kirillov's theorem applied to $(n-2)$-order hereditary structures of the considered algebra. □

Recall that usual Lie algebra structures defined by means of first order bi-differential operators are called *Jacobi*'s [11,12,16]. This motivates the following terminology.

Definition 5.2. An n-Jacobi manifold (structure) is a manifold M supplied with a local n-Lie algebra structure on $C^\infty(M)$ given by a first order n-differential operator.

Hence, in these terms Proposition 5.1 says that multi-Jacobi structures exhaust local multi-Lie algebra ones. Note, however, that it seems not to be the case for infinite-dimensional manifolds that occur in secondary calculus. Kirillov gives also an exhaustive description of Jacobi manifolds.

Namely, Kirillov showed that a binary Jacobi bracket $[\cdot\,,\,\cdot]$ on a manifold M can be uniquely presented in the form

$$[f, g] = T(\mathrm{d}f, \mathrm{d}g) + fX(g) - gX(f)$$

with X and T being a vector field and a bivector field, respectively, such that $\lceil T, T \rfloor = X \wedge T$ and $L_X(T) = 0$. Then two qualitatively different situations can occur: $X \wedge T \equiv 0$ and $X \wedge T \neq 0$ (locally). In the first of them the bivector T is a Poisson one of rank 0 or 2. In the latter case X is a locally Hamiltonian field with respect to T, i.e. $X = T_f$

for an appropriate $f \in C^\infty(M)$. If $X \wedge T \neq 0$, then M is foliated (with singularities) by $(2n+1)$-dimensional leaves with $2n = \operatorname{rank} T$ (an analog of the Hamiltonian foliation) and the original Jacobi structure is reduced to a family of locally contact brackets [12,16] on leaves of this foliation.

Below we find an n-ary analog of Kirillov's theorem for $n > 2$ showing that in this case only the first possibility of the two mentioned above survives. Fundamental here is a canonical decomposition of the first order skew-symmetric multi-differential operator Δ defining the local n-Lie algebra in question which we are passing to describe.

Recall, first, that a first order linear (scalar) differential operator on M is a $\mathbb{R}$-linear map $\nabla : C^\infty(M) \to C^\infty(M)$ such that

$$\nabla(fg) = f\nabla(g) + g\nabla(f) - fg\nabla(1) \quad \forall f, g \in C^\infty(M). \tag{5.1}$$

This algebraic definition is equivalent to the standard coordinate one [12]. It characterizes vector fields on M, i.e. derivations of $C^\infty(M)$, as first order differential operators ∇ such that $\nabla(1) = 0$. Let Δ be a skew-symmetric first order n-differential operator. According to the adopted notation Δ_1 is an $(n-1)$-differential operator defined as $\Delta_1(f_1, \ldots, f_{n-1}) = \Delta(1, f_1, \ldots, f_{n-1})$. Obviously, it is of first order. Moreover, it is a multi-derivation. In fact, it is seen immediately from what was said before by observing that owing to skew-commutativity

$$\Delta_1(1, \ldots) = (\Delta_1)_1 = \Delta_{1,1} = 0. \tag{5.2}$$

If Γ is a skew-symmetric k-derivation, then the $(k+1)$-differential operator $s(\Gamma)$ defined as

$$s(\Gamma)(f_1, \ldots, f_{k+1}) = \sum_i (-1)^{i-1} f_i \Gamma(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{k+1}) \tag{5.3}$$

is, obviously, skew-symmetric and of first order. Moreover, $s(\Gamma)_1 = \Gamma$. By applying this construction to $\Gamma = \Delta_1$ we obtain the first order skew-symmetric n-differential operator $\Delta^0 = s(\Delta^1)$ such that $(\Delta^0)_1 = \Delta_1$. Last relation shows that the n-differential operator $\hat{\Delta} = \Delta - \Delta^0$ is an n-derivation. Now gathering together what was done before we obtain:

Proposition 5.2. *With any first order skew-symmetric n-differential operator Δ are associated skew-symmetric multi-derivations $\hat{\Delta}$ and Δ_1 of multiplicities n and $n-1$, respectively, such that (canonicaldecomposition)*

$$\Delta = \hat{\Delta} + \Delta^0 \tag{5.4}$$

with $\Delta^0 = s(\Delta_1)$, i.e.

$$\Delta^0(f_1, \ldots, f_n) = \sum_i (-1)^{i-1} f_i \Delta_1(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n). \tag{5.5}$$

Conversely, any pair (∇, Γ) of skew-symmetric derivations of multiplicities n and $n-1$, respectively, defines a unique skew-symmetric n-differential operator of first order $\Delta = \nabla + s(\Gamma)$ such that $\nabla = \hat{\Delta}$ and $\Gamma = \Delta_1$.

It is natural to extend the operation s from the skew-symmetric derivations to arbitrary skew-symmetric multi-differential operators. Namely, if Δ is a skew-symmetric k-differential operator, then we put

$$s(\Delta)(g_1, \dots, g_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} g_i \Delta(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{k+1}).$$

This way we get the map

$$s : \mathit{Diff}^{alt}_{l|k}(M) \to \mathit{Diff}^{alt}_{l|k+1}(M),$$

$\mathit{Diff}^{alt}_{l|k}(M)$ denoting the space of lth order $C^\infty(M)$-valued skew-symmetric k-differential operators on $C^\infty(M)$.

Proposition 5.3. *The operation s is $C^\infty(M)$-linear and $s^2 = 0$.*

Proof. Obvious. □

Remark 5.1. Proposition 5.3 shows that s can be viewed as the differential of the complex

$$0 \to \mathit{Diff}^{alt}_{l|1}(M) \xrightarrow{s} \mathit{Diff}^{alt}_{l|2}(M) \xrightarrow{s} \cdots \xrightarrow{s} \mathit{Diff}^{alt}_{l|k}(M) \xrightarrow{s} \cdots$$

This complex is acyclic in positive dimensions and its 0-cohomology group is isomorphic to $C^\infty(M)$. In fact, the insertion of the unity operator i_1 is a homotopy operator for s as it results from Proposition 5.2.

Further properties of s we need are the following.

Proposition 5.4. *The operation s has the properties:*
(1) *If $X \in D(M)$, then $[L_X, s] = 0$.*
(2) *If $f \in C^\infty(M)$, then $f \rfloor s(\Box) + s(f \rfloor \Box) = f\Box$.*
(3) $s(\Box)_{f_1,\dots,f_k} = \sum_i (-1)^{i-1} f_i \Box_{f_1,\dots,f_{i-1},f_{i+1},\dots,f_k} + (-1)^k \Box(f_1, \dots, f_k)$.

Proof. We start with (1).
For $\Box \in \mathit{Diff}^{alt}_{l|k}(M)$ one has by definition

$$L_X(s(\Box))(g_1, \dots, g_{k+1}) = X(s(\Box)(g_1, \dots, g_{k+1})) - \sum_i s(\Box)(g_1, \dots, X(g_i), \dots, g_{k+1}).$$

But

$$X(s(\Box)(g_1, \dots, g_{k+1})) = \sum_i (-1)^{i-1} X(g_i)\Box(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{k+1}) + \sum_i (-1)^{i-1} g_i X(\Box(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{k+1})$$

and

$$
\begin{aligned}
&s(\Box)(g_1,\dots,X(g_i),\dots,g_{k+1})\\
&\quad=(-1)^{i-1}X(g_i)\Box(g_1,\dots,g_{i-1},g_{i+1},\dots,g_{k+1})\\
&\qquad+\sum_{j<i}(-1)^{j-1}g_j\Box(g_1,\dots,g_{j-1},g_{j+1},\dots,X(g_i),\dots,g_{k+1})\\
&\qquad+\sum_{i<j}(-1)^{j-1}g_j\Box(g_1,\dots,X(g_i),\dots,g_{j-1},g_{j+1},\dots,g_{k+1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&L_X(s(\Box))(g_1,\dots,g_{k+1})\\
&\quad=\sum_i(-1)^{i-1}g_iX(\Box(g_1,\dots,g_{i-1},g_{i+1},\dots,g_{k+1})\\
&\qquad+\sum_{j<i}(-1)^{j-1}g_j\Box(g_1,\dots,g_{j-1},g_{j+1},\dots,X(g_i),\dots,g_{k+1})\\
&\qquad+\sum_{i<j}(-1)^{j-1}g_j\Box(g_1,\dots,X(g_i),\dots,g_{j-1},g_{j+1},\dots,g_{k+1})\\
&\quad=\sum_i(-1)^{i-1}g_iX(\Box)(g_1,\dots,g_{i-1},g_{i+1},\dots,g_{k+1})\\
&\quad=s(L_X(\Box))(g_1,\dots,g_{k+1}).
\end{aligned}
$$

Thus, $s\circ L_X=L_X\circ s\Leftrightarrow[L_X,s]=0$.

Property (2) is an immediate consequence of the definition of s . Finally, (3) is obtained from (2) by an obvious induction. □

We need also the following formula concerning Lie derivative.

Lemma 5.1. *If $f\in C^\infty(M)$ and $\Box$ is a skew-symmetric k-derivation, then*

$$L_f(\Box)=(1-k)f\Box-s(\Box_f),$$

where the Lie derivative L_f is understood in the sense of Section 2.

Proof. By definition

$$
\begin{aligned}
&L_f(\Box)(g_1,\dots,g_k)\\
&\quad=f\Box((g_1,\dots,g_k)-\sum\Box(g_1,\dots,fg_i,\dots,g_k)\\
&\quad=f\Box(g_1,\dots,g_k)\\
&\qquad-\sum_i(f\Box(g_1,\dots,g_k)+g_i\Box(g_1,\dots,g_{i-1},f,g_{i+1},\dots,g_k)\\
&\quad=(1-k)f\Box(g_1,\dots,g_k)+\sum(-1)^{i-1}g_i\Box(f,g_1,\dots,g_{i-1},g_{i+1}\dots,g_k).
\end{aligned}
$$

But last summation coincides, obviously, with $-s(\Box_f)(g_1,\dots,g_k)$. □

Proposition 5.2 suggests to treat the problem of describing n-Jacobi structures as determination of conditions to impose on a pair of multi-derivations ∇ and $\Box$ of multiplicities n and $n-1$, respectively, in order that the n-differential operator $\Delta = \nabla + s(\Box)$ be an n-Jacobi one. In other words, we have to resolve the equation

$$(\nabla + s(\Box))_{f_1,\dots,f_{n-1}}(\nabla + s(\Box)) = 0 \tag{5.6}$$

with respect to ∇ and $\Box$. So we pass to analyze Eq. (5.6).

First, by applying Proposition 5.4, (3) and posing $X_i = \Box_{f_1,\dots,\hat{f}_i,\dots,f_{n-1}}$ and $h = (-1)^{n-1}\Box(f_1,\dots,f_{n-1})$ one finds

$$s(\Box)_{f_1,\dots,f_{n-1}}(\nabla) = \sum(-1)^{i-1}(f_iX_i)(\nabla) + L_h(\nabla).$$

The following expression is computed with the help of Lemmas 4.5 and 5.1:

$$s(\Box)_{f_1,\dots,f_{n-1}}(\nabla) = \sum(-1)^{i-1}(f_iX_i(\nabla) - X_i \wedge \nabla_{f_i}) + (1-n)h\nabla - s(\nabla_h). \tag{5.7}$$

Similarly, taking into account Proposition 5.4 (1), Lemmas 4.5 and 5.1 and the fact that $s(\Box)_{f_1,\dots,f_{n-1}} = Y + h$ with $Y = \sum(-1)^{i-1}f_iX_i \in \mathcal{D}(M)$ one finds

$$\begin{aligned} &s(\Box)_{f_1,\dots,f_{n-1}}(s(\Box)) \\ &\quad = (Y+h)(s(\Box)) = s(Y(\Box)) + (1-n)hs(\Box) \\ &\quad = s\Big(\sum(-1)^{i-1}f_iX_i(\Box) - \sum(-1)^{i-1}X_i \wedge \Box_{f_i} + (1-n)h\Box\Big). \end{aligned} \tag{5.8}$$

Putting together formulae (5.7) and (5.8) we obtain the key technical result of this section.

Proposition 5.5. *Let ∇ and $\Box$ be skew-symmetric multi-derivations of multiplicity n and $n-1$, respectively, then the canonical decomposition of the skew-symmetric k-differential operator*

$$(\nabla + s(\Box))_{f_1,\dots,f_{n-1}}(\nabla + s(\Box))$$

is given by the formula

$$(\nabla + s(\Box))_{f_1,\dots,f_{n-1}}(\nabla + s(\Box)) = \Delta^1 + s(\Delta^0),$$

where

$$\begin{aligned} \Delta^1(f_1,\dots,f_{n-1}) = {} & \nabla_{f_1,\dots,f_{n-1}}(\nabla) \\ & + \sum_{i=1}^{n-1}(-1)^{i-1}(f_iX_i(\nabla) - X_i \wedge \nabla_{f_i}) + (1-n)h\nabla \end{aligned}$$

and

$$\Delta^0(f_1,\ldots,f_{n-1}) = \nabla_{f_1,\ldots,f_{k-1}}(\Box) - \nabla_h + \sum_{i=1}^{n-1}(-1)^{i-1}(f_i X_i(\Box) - X_i \wedge \Box_{f_i}) + (1-n)h\Box$$

with $X_i = \Box_{f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_{n-1}}$, $h = (-1)^{n-1}\Box(f_1,\ldots,f_{n-1})$.

Corollary 5.1. *If* $\Delta + s(\Box)$ *is n-Jacobian, then for any* $g_1,\ldots,g_{n-2} \in C^\infty(M)$

$$\Box_{g_1,\ldots,g_{n-2}}(\nabla) = 0 \quad \textit{and} \quad \Box_{g_1,\ldots,g_{n-2}}(\Box) = 0$$

In particular, $\Box$ *is an* $(n-1)$*-Poisson structure.*

Proof. In virtue of Proposition 5.5 Eq. (5.6) is equivalent to

$$\Delta^0(f_1,\ldots,f_{n-1}) = 0, \qquad \Delta^1(f_1,\ldots,f_{n-1}) = 0.$$

It remains to note that

$$\Delta^0(1,g_1,\ldots,g_{n-2}) = \Box_{g_1,\ldots,g_{n-2}}(\Box),$$
$$\Delta^1(1,g_1,\ldots,g_{n-2}) = \Box_{g_1,\ldots,g_{n-2}}(\nabla). \qquad \Box$$

Put

$$\Delta_0^1(f_1,\ldots,f_{n-1}) := \nabla_{f_1,\ldots,f_{n-1}}(\nabla) + \sum_{i=1}^{n-1}(-1)^{i-1} f_i \rfloor (X_i \wedge \nabla),$$
$$\Delta_0^0(f_1,\ldots,f_{n-1}) := \nabla_{f_1,\ldots,f_{n-1}}(\Box) + \sum_{i=1}^{n-1}(-1)^{i-1} f_i \rfloor (X_i \wedge \Box) - \nabla_h.$$

Corollary 5.2. *If* $\nabla + s(\Box)$ *is an n-Jacobian, then*

$$\Delta_0^0(f_1,\ldots,f_{n-1}) = 0 \quad \text{and} \quad \Delta_0^1(f_1,\ldots,f_{n-1}) = 0.$$

Proof. Corollary 5.1 shows that $X_i(\Box) = 0$ and $X_i(\nabla) = 0$. Also we have

$$\begin{aligned}(-1)^{i-1} f_i \rfloor X_i &= (-1)^{i-1} f_i \rfloor \Box_{f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_{n-1}} \\ &= (-1)^{i-1}\Box(f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_{n-1},\ldots,f_i) \\ &= (-1)^{n-1}\Box(f_1,\ldots,f_{n-1}) = h.\end{aligned}$$

Hence,

$$-\sum_{i=1}^{n-1}(-1)^{i-1} X_i \wedge \nabla_{f_i} + (1-n)h\nabla = \sum_{i=1}^{n-1}(-1)^{i-1} f_i \rfloor (X_i \wedge \nabla).$$

and

$$-\sum_{i=1}^{n-1}(-1)^{i-1} X_i \wedge \Box_{f_i} + (1-n)h\Box = \sum_{i=1}^{n-1}(-1)^{i-1} f_i \rfloor (X_i \wedge \Box). \qquad \Box$$

Proposition 5.6. *$(n-1)$-differential operators Δ_0^0 and Δ_0^1 satisfy relations*

$$\begin{aligned}\Delta_0^0(\varphi\psi, g_1, \dots, g_{n-2}) = {} & \varphi\Delta_0^0(\psi, g_1, \dots, g_{n-2}) + \psi\Delta_0^0(\varphi, g_1, \dots, g_{n-2}) \\ & - \nabla_{\varphi, g_1, \dots, g_{n-2}} \wedge \Box_\psi - \nabla_{\psi, g_1, \dots, g_{n-2}} \wedge \Box_\varphi \\ & - (-1)^{n-1}\Box(\varphi, g_1, \dots, g_{n-2})\nabla_\psi \\ & - (-1)^{n-1}\Box(\psi, g_1, \dots, g_{n-2})\nabla_\varphi \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\Delta_0^1(\varphi\psi, g_1, \dots, g_{n-2}) = {} & \varphi\Delta_0^1(\psi, g_1, \dots, g_{n-2}) + \psi\Delta_0^1(\varphi, g_1, \dots, g_{n-2}) \\ & - \nabla_{\varphi, g_1, \dots, g_{n-2}} \wedge \nabla_\psi - \nabla_{\psi, g_1, \dots, g_{n-2}} \wedge \nabla_\varphi. \end{aligned} \tag{5.10}$$

Proof. This is essentially the same as the proof of Proposition 4.2. One has to make use of the fact that the maps $f \longmapsto \nabla_f$ and $f \longmapsto \Box_f$ are derivations and to apply Lemma 4.5. □

Corollary 5.3. *If $\nabla + s(\Box)$ is n-Jacobian, then*

$$\begin{aligned}& \nabla_{\varphi, g_1, \dots, g_{n-2}} \wedge \Box_\psi + \nabla_{\psi, g_1, \dots, g_{n-2}} \wedge \Box_\varphi + (-1)^{n-1}[\Box(\varphi, g_1, \dots, g_{n-2})\nabla_\psi \\ & \quad + \Box(\psi, g_1, \dots, g_{n-2})\nabla_\varphi] = 0 \end{aligned} \tag{5.11}$$

and

$$\nabla_{\varphi, g_1, \dots, g_{n-2}} \wedge \nabla_\psi + \nabla_{\psi, g_1, \dots, g_{n-2}} \wedge \nabla_\varphi = 0. \tag{5.12}$$

Proof. Immediate from formulae (5.9) and (5.10) and Corollary 5.2. □

Corollary 5.4. *If $\nabla + s(\Box)$ is n-Jacobian, then the n-vector, corresponding to ∇ is locally either of rank n (i.e. locally decomposable) for $n > 2$, or trivial.*

Proof. Observe that Theorem 4.1 results from formula (4.6) which is identical to (5.12). □

Denote by V and W multi-vectors corresponding to ∇ and $\Box$, respectively. Let Π_x and P_x, $x \in M$, be subspaces of T_xM generated by derived vectors of V_x and W_x, respectively.

Proposition 5.7. *If $\nabla + s(\Box)$ is n-Jacobian with $n > 2$, then $\mathrm{rank}(W_x) \leq n-1$ and $P_x \subset \Pi_x$ if $V_x \neq 0$.*

Proof. Relation $\Box_{g_1, \dots, g_{n-2}}(\nabla) = 0$ (Corollary 5.1) implies

$$\Box_{\varphi, g_1, \dots, g_{n-3}} \wedge \nabla_\psi + \Box_{\psi, g_1, \dots, g_{n-3}} \wedge \nabla_\varphi = 0. \tag{5.13}$$

This can be proved repeating literally the reasoning used above to deduce formula (4.6). In terms of multi-vectors relation (5.13) is equivalent to

$$(\mathrm{d}g_{n-3}\rfloor\cdots\rfloor \mathrm{d}g_1\rfloor \mathrm{d}\varphi\rfloor W)\wedge(\mathrm{d}\psi\rfloor V)+(\mathrm{d}g_{n-3}\rfloor\cdots\rfloor \mathrm{d}g_1\rfloor \mathrm{d}\psi\rfloor W)\wedge(\mathrm{d}\varphi\rfloor V)=0. \tag{5.14}$$

In particular, for $\varphi=\psi$ we have

$$(\mathrm{d}g_{n-3}\rfloor\cdots\rfloor \mathrm{d}g_1\rfloor \mathrm{d}\varphi\rfloor W)\wedge(\mathrm{d}\varphi\rfloor V). \tag{5.15}$$

By Lemma 4.2 (5.15) shows that the derived vector

$$(\mathrm{d}g_{n-3}\rfloor\cdots\rfloor \mathrm{d}g_1\rfloor \mathrm{d}\varphi\rfloor W)$$

divides $\mathrm{d}\varphi\rfloor V$. Since V is of rank n it divides also V. This proves the inclusion $P_x\subset \Pi_x$.

Further, being $W(n-1)$-Poissonian (Corollary 5.1) $rank(W)\le n-1$ if $n>3$. For $n=3$ the inclusion $P_x\subset\Pi_x$ shows that $rank(W)\le 3$ due to decomposability of V. But the rank of a bivector is an even number. So, $rank(W)\le 2$. □

Corollary 5.5. *If $\nabla+s(\Box)$ is n-Jacobian with $n>2$, then $X_i\wedge\nabla=0$ and $X_i\wedge\Box=0$.*

Proof. X_i is a derived vector of W and, due to inclusion $P_x\subset\Pi_x$, is also a derived vector of V. It remains to observe that a decomposable multi-vector vanishes when being multiplied by any of its derived vectors. □

Corollary 5.6. *If $\nabla+s(\Box)$ is n-Jacobian with $n>2$, then*

$$\nabla_{f_1,\ldots,f_{n-1}}(\nabla)=0, \tag{5.16}$$

$$\nabla_{f_1,\ldots,f_{n-1}}(\Box)=\nabla_h. \tag{5.17}$$

In particular, ∇ is an n-Poisson structure on M.

Proof. Immediate from Corollary 5.2. □

Below it is supposed that $\Delta=\nabla+s(\Box)$ defines an n-Jacobi structure on M with $n>2$. A point $x\in M$ of that n-Poisson manifold is called *regular* if both multi-vectors V and W corresponding to ∇ and $\Box$, respectively, do not vanish at x. Note that the inclusion $P_x\subset\Pi_x$ (Proposition 5.7) implies that x is regular if $\Box$ is regular at x , i.e. $W_x\neq 0$.

Now we can prove the main structural result concerning n-Jacobian manifolds with $n>2$.

Theorem 5.1. *Let Δ be a non-trivial n-Jacobi structure and $n>2$. Then in a neighborhood of any of its regular points it is either of the form $\Delta=\nabla+s(\nabla_h)$ where ∇ is a non-trivial n-Poisson structure, or $\Delta=s(\Box)$ where $\Box$ is an $(n-1)$-Poisson structure (of rank 2 if n=3).*

Proof. Corollary 5.1 and Proposition 5.7 show that $\Box$ is an $(n-1)$-Poisson structure of rank $\le n-1$ on M while Corollaries 5.4 and 5.6 show ∇ to be an n-Poisson one of rank n.

Hamiltonian foliations of these two multi-Poisson structures (we call them $\Box$-foliaton and ∇-foliation, respectively) are regular foliations of dimensions $n-1$ and n, respectively, in a neighborhood of a regular point $a \in M$. Moreover, $\Box$-foliation is inscribed into ∇-foliation according to Proposition 5.7. So, if the neighborhood $\mathcal{U}$ of a is sufficiently small, there exist a system of functionally independent functions $y, z_1, \ldots, z_{m-n}$, $m = \dim M$ such that they all are constant along leaves of the $\Box$-foliation and $z_1, \ldots, z_{m-n}$ are constant along leaves of the ∇-foliation.

Since $\Box$ is $(n-1)$-Poisson of rank $n-1$ there exist (locally) mutually commuting vector fields $X_1, \ldots, X_{n-1}$ such that $\Box = X_1 \wedge \cdots \wedge X_{n-1}$. We can assume that $X_i \in D(\mathcal{U})$. Then it is easy to see that there exist functions $x_1, \ldots, x_{n-1} \in C^\infty(\mathcal{U})$ such that $X_i(x_j) = \delta_{ij}$. Vector fields X_i's are, obviously, tangent to leaves of $\Box$-foliation and, therefore, $X_i(y) = X_i(z_j) = 0\ \forall j$. By construction functions $x_1, \ldots, x_{n-1}, y, z_1, \ldots, z_{m-n}$ are independent (functionally). So they form a local chart in $\mathcal{U}$ in, maybe, smaller neighborhood of a. Now vector fields X_i's are identified with $(\partial/\partial x_i)$'s, partial derivations in the sense of the above local chart. Note also, that the vector field $\partial/\partial y$ is tangent to leaves of ∇-foliation. By construction the n-vector V is tangent also to these leaves. By this reason

$$\nabla = \lambda \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{n-1}}$$

with $\lambda \in C^\infty(\mathcal{U})$.

Observe now that $\partial/\partial x_i$ is a $\Box$-Hamiltonian vector field associated with the Hamiltonian $((-1)^{i-1}x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})$.

For this field relation $\Box_{g_1,\ldots,g_{n-2}}(\nabla) = 0$ (Corollary 5.1) becomes

$$\frac{\partial}{\partial x_i}\left(\lambda \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_{n-1}}\right) = 0, \tag{5.18}$$

which is equivalent to $\partial\lambda/\partial x_i = 0$. This shows that $\lambda = \lambda(y, z_1, \ldots, z_{m-n})$. Hence, vector fields $X_1 = \partial/\partial x_1, \ldots, X_{n-1} = \partial/\partial x_{n-1}$, $X_n = \lambda(\partial/\partial y)$ commute and, therefore, there exist functions $y_1, \ldots, y_n \in C^\infty(\mathcal{U})$ such that $X_i(y_i) = \delta_{ij}$, $\quad i, j = 1, \ldots, n$. Obviously, functions $y_1, \ldots, y_n, z_1, \ldots, z_{m-n}$ constitute a local chart with respect to which $X_i = \partial/\partial y_i$, $i = 1, \ldots, n$. Thus, we have proved that

$$\nabla = \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}, \qquad \Box = \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_{n-1}}. \tag{5.19}$$

It remains to note that $\Box = \nabla_h$ for $h = (-1)^{n-1} y_n$. This proves the first part of the theorem.

To prove the second one we observe that if $\nabla \equiv 0$ in the canonical decomposition of Δ, i.e. $\Delta = s(\Box)$, Corollaries 5.1 and 5.5 show that $\Box$ is an $(n-1)$-Poisson structure of rank $n-1$. (In virtue of Theorem 4.1 last condition is essential only if $n = 3$.) On the other hand, one can see easily that when $\nabla \equiv 0$ any such Poisson structure satisfies conditions $\Delta^1 = 0$, $\Delta^0 = 0$ of Proposition 5.5. $\Box$

Corollary 5.7. *If M is an n-Jacobian manifold and $n > 2$, then in a neighborhood of its regular point a local chart $y_1, \ldots, y_n, z_1, \ldots, z_{m-n}$ exists such that*

$$\{f_1, \ldots, f_n\} = \det \left\| \frac{\partial f_i}{\partial y_j} \right\| + \sum_{k=1}^{n} (-1)^{k-1} f_k \det \left\| \frac{\partial f_i}{\partial y_j} \right\|_k ,$$

where $\|\partial f_i / \partial f_j\|_k$ is the $(n-1) \times (n-1)$-matrix obtained from the $n \times n$-matrix $\|\partial f_i / \partial y_j\|$ by canceling its kth row and nth column.

Proof. It results directly from (5.19) and the definition of s. □

Proposition 5.8. *Let ∇ be an n-Poisson structure of rank n on M and $f \in C^\infty(M)$. Then $\Delta = \nabla + s(\nabla_f)$ is an n-Jacobi structure. In particular, this is the case for any n-Poisson ∇ with $n > 2$.*

Proof. With the notation of Proposition 5.5 $X_i = \nabla_{f, f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_{n-1}}$ and $\Box = \nabla_f$. By this reason $X_i(\nabla) = 0$ as well as $\nabla_{f_1, \ldots, f_{n-1}}(\nabla) = 0$.

Therefore, the n-differential operator $\Delta^1(f_1, \ldots, f_{n-1})$ (Proposition 5.5) is reduced to $\sum_{i=1}^{n-1} (-1)^{i-1} f_i \rfloor (X_i \wedge \nabla)$. Moreover, $X_i \wedge \nabla = 0$ due to the fact that ∇ is of rank n. Hence, in the considered context $\Delta^1(f_1, \ldots, f_{n-1}) = 0$.

Next, $X_i(\nabla_f) = 0$ since ∇_f is an $(n-1)$-Poisson structure.

By applying formula (2.8) for $\delta = \nabla_{f_1, \ldots, f_{n_1}}$ and $u = f$ we see that for $h = (-1)^{n-1} \Box (f_1, \ldots, f_{n-1}) = (-1)^{n-1} \nabla(f, f_1, \ldots, f_{n-1})$

$$\nabla_{f_1, \ldots, f_{n-1}}(\nabla_f) - \nabla_h = f \rfloor \nabla_{f_1, \ldots, f_{n-1}}(\nabla) = 0.$$

So, the $(n-1)$-differential operator $\Delta^0(f_1, \ldots, f_{n-1})$ (Proposition 5.5) is reduced to $\sum_{k=1}^{n-1} (-1)^{i-1} f_i \rfloor (X_i \wedge \nabla_f)$. But ∇_f is obviously, of rank $\leq n-1$ and so $X_i \wedge \nabla_f = 0$. Hence, $\Delta^0(f_1, \ldots, f_{n-1}) = 0$. It proves that Δ is n-Jacobian. □

The construction of Proposition 5.8 can be generalized as follows. Let ω be a closed differential form of order 1. For a multi-derivation ∇ define another one ∇^ω by putting locally $\nabla^\omega = \nabla_f$ if $\omega = \mathrm{d}f$. This definition is, obviously, correct and allows to globalize Proposition 5.8.

Proposition 5.9. *If ∇ is an n-Poisson structure of rank n, then $\Delta = \nabla + s(\nabla^\omega)$ is an n-Jacobi structure for any closed differential 1-form ω.*

Proof. It results directly from Proposition 5.8 and from the fact that the n-Jacobi identity for Δ is a multi-differential operator. □

Example 5.2. With notation of example 3.2 consider the $(n+1)$-Poisson structure

$$\nabla = \frac{\partial}{\partial \theta_1} \wedge \cdots \wedge \frac{\partial}{\partial \theta_{n+1}}$$

on $(n+1)$-torus T^{n+1}. Then ∇^{ω} with the closed but not exact on T^{n+1} 1-form $\omega = \alpha\, d\theta_1 - d\theta_2$ gives the n-Poisson structure described in Example 3.2. Therefore the $(n+1)$-Jacobi structure $\Delta = \nabla + s(\nabla^{\omega})$ on T^{n+1} is such that the leaves of its $\Box$-*foliation* are everywhere dense in the unique leaf, T^{n+1}, of its ∇-*foliation*.

It is not difficult to show that any n-Jacobi strucure with $n > 2$ on an n-dimensional manifold is of the form $\nabla + s(\nabla^{\omega})$ for suitable closed 1-form ω and n-Poisson structure ∇ on M.

6. n-Bianchi classification

In view of the conjecture of Section 3 on the structure of n-Lie algebras for $n > 2$ a classification of $(n+1)$-dimensional n-Lie algebras turns out to be of a particular interest. Such a classification, an analog of that of Bianchi for three-dimensional Lie algebras, is, in fact, already done in [7] by a direct algebraic approach. Below we get it in a transparent geometric way which, in addition, reveals some interesting peculiarities.

To start with, observe that on an orientable $(n+1)$-dimensional manifold M any n-vector P can be given in the form

$$P = \alpha \rfloor V$$

with a 1-form $\alpha = \alpha_{P,V}$ and a (prescribed) volume $(n+1)$-vector field V on M, respectively. Obviously, $\alpha \rfloor P = 0$. This means that α vanishes on the n-dimensional distribution defined by P.

If P is an n-Poisson one, this distribution is tangent to the corresponding Hamiltonian foliation and as such is integrable. Therefore, $\alpha \wedge d\alpha = 0$. In virtue of Proposition 3.2 this condition is sufficient for P to be an n-Poisson vector field.

Let us call an n-Poisson structure *unimodular* with respect to V if for any n-Hamiltonian vector field X $L_X(V) = 0$.

Proposition 6.1. *An n-Poisson structure P is V-unimodular iff* $d\alpha_{P,V} = 0$.

Proof. Recall the general formula

$$L_X(\alpha \rfloor V) = \alpha \rfloor L_X(V) - L_X(\alpha) \rfloor V, \tag{6.1}$$

which holds for arbitrary vector field X, differential form α and multi-vector field V. If X is a P-Hamiltonian field with $P = \alpha \rfloor V$, then $L_X(\alpha \rfloor V) = 0$ and (6.1) gives

$$\alpha \rfloor L_X(V) = L_X(\alpha) \rfloor V.$$

Since, also, $X \rfloor \alpha = 0$, $L_X(\alpha) = X \rfloor d\alpha$ and the last equality can be rewritten as

$$\mathrm{div}_V X \cdot P = (X \rfloor d\alpha) \rfloor V \tag{6.2}$$

due to the fact that $L_X(V) = \mathrm{div}_V X \cdot V$. So, $\mathrm{div}_V X = 0 \Leftrightarrow L_X(V) = 0$ for any P-Hamiltonian field X if $d\alpha = 0$.

Conversely, (6.2) shows that $X \rfloor \mathrm{d}\alpha$ vanishes for any P-Hamiltonian field X if P is V-unimodular. This implies that $Y \rfloor \mathrm{d}\alpha = 0$ for any Y tangent to the Hamiltonian foliation of P. Since this foliation is of codimension 1 any decomposable bivector B on M can be presented at least locally, in form $B = Z \wedge Y$ with Y as above. This shows that $B \rfloor \mathrm{d}\alpha = 0$ for any decomposable B and, hence, $\mathrm{d}\alpha = 0$. □

Now we specify the above construction to the case $M = \mathcal{V}^*, \mathcal{V}$ being an $(n+1)$-dimensional vector space and $P = T$, T being the n-Poisson structure on $\mathcal{V}^*$ associated with an n-Lie algebra structure on $\mathcal{V}$. Also, we consider the $(n+1)$-vector field

$$V = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{n+1}}$$

on $\mathcal{V}$ where x_i's are some cartesian coordinates on $\mathcal{V}^*$. Such an $(n+1)$-field is defined uniquely up to a scalar factor. So, the above concept of unimodularity does not depend on the choice of such a V and the 1-form $\alpha_{T,V}$ is defined uniquely up to a scalar factor. Note also that $\alpha_{T,C}$ is *linear* in the sense that the function $\Xi \rfloor \alpha_{T,C}$ is linear on $\mathcal{V}^*$, i.e. an element of $\mathcal{V}$, for any constant vector field Ξ. In coordinates this means that $\alpha_{T,V}$ looks as

$$\alpha_{T,V} = \sum_{i,j} a_{ij} x_j \, \mathrm{d}x_i, \quad a_{ij} \in \mathbb{R}.$$

Proposition 6.2. *Algebraic variety of n-Lie algebra structures on $\mathcal{V}$ is identical to the variety of linear differential 1-forms on $\mathcal{V}^*$ satisfying the condition $\alpha \wedge \mathrm{d}\alpha = 0$.*

Proof. It was already shown that any n-Lie algebra structure on $\mathcal{V}$ is characterized uniquely by the corresponding linear differential 1-form $\alpha_{T,C}$.

Conversely, if α is a linear differential 1-form, then n-ary operation on $C^\infty(\mathcal{V}^*)$ defined by the n-vector field $\alpha \rfloor V$ is closed on the subspace of linear functions on $\mathcal{V}^*$, i.e. on $\mathcal{V}$. This way one gets an n-ary operation on V. The condition $\alpha \wedge \mathrm{d}\alpha = 0$ guarantees integrability of the n-distribution on $\mathcal{V}^*$ defined by $P = \alpha \rfloor V$ and by virtue of Corollary 3.2 it is an n-Poisson structure. This fact restricted on V shows the above n-ary operation to be an n-Lie one. □

Note now that any linear differential 1-form on $\mathcal{V}^*$ can be identified with a bilinear 2-form b on $\mathcal{V}^*$. Namely, denote by C_ω the constant field of vectors on $\mathcal{V}^*$ which are equal to $\omega \in \mathcal{V}^*$ and put

$$b(\omega, \rho) := (C_\omega \rfloor \alpha, \rho), \quad \omega, \rho \in \mathcal{V}^*,$$

where bracket $(\cdot, \cdot)$ stands for a natural pairing of $\mathcal{V}$ and $\mathcal{V}^*$. Obviously,

$$b(\omega, \rho) = \sum_{i,j} a_{i,j} \omega_i \rho_j$$

if $\omega = \sum \omega_i (\partial/\partial x_i)$, $\rho = \sum \rho_j (\partial/\partial x_j)$ and $\alpha = \sum a_{ij} x_j \, \mathrm{d}x_i$. So, $\|a_{ij}\|$ is the matrix of b. The form b is called *generating* for the n-Lie algebra in question.

An n-Lie algebra is called *unimodular* if all its inner derivations are unimodular operators. For an $(n+1)$-dimensional n-Lie algebra this is, obviously, equivalent to unimodularity of the associated Poisson structure T on $\mathcal{V}^*$ with respect to a cartesian volume $(n+1)$-vector V. On the other hand, T is V-unimodular iff $\mathrm{d}\alpha_{T,V} = 0$ (Proposition 6.1) and for a linear differential 1-form α the condition $\mathrm{d}\alpha = 0$ is equivalent to $\alpha = \mathrm{d}F$ for a quadratic polynomial F on $\mathcal{V}^*$ (or to symmetry of the corresponding bilinear form b). These considerations prove the following result.

Proposition 6.3. *The n-Poisson structure T on $\mathcal{V}^*$ associated with a unimodular Lie algebra structure on an $(n+1)$-dimensional vector space $\mathcal{V}$ is of the form $\mathrm{d}F \rfloor V$ for a suitable quadratic polynomial F on $\mathcal{V}^*$. Therefore, all unimodular n-Lie structures on $\mathcal{V}$ are mutually compatible. Two such structures are isomorphic iff the corresponding quadratic polynomials can be reduced to one another up to a scalar factor by a linear transformation. In particular, for $k = \mathbb{R}$ isomorphic classes of unimodular $(n+1)$-dimensional n-Lie structures can be labeled by two numbers: r (the rank of F), $0 \le r \le n+1$ and m (the maximal of positive and negative indices of F), $\frac{1}{2}r \le m \le r$.*

Passing now to the case $\mathrm{d}\alpha_{T,V} \neq 0$ we note that $\mathrm{d}\alpha_{T,V}$ is a constant differential 2-form on $\mathcal{V}^*$ due to linearity of $\alpha_{T,V}$. Moreover, the condition $\alpha_{T,V} \wedge \mathrm{d}\alpha_{T,V} = 0$ shows that the rank of $\mathrm{d}\alpha_{T,V}$ is equal to 2. Therefore, $\mathrm{d}\alpha_{T,V} = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ in suitable cartesian coordinates on $\mathcal{V}^*$. Since $\alpha_{T,V}$ divides $\mathrm{d}x_1 \wedge \mathrm{d}x_2$ and is linear it must be of the form

$$\sum_{i=1}^{2} \mu_{ij} x_j \,\mathrm{d}x_i \qquad \text{with } \mu_{21} - \mu_{12} = 1.$$

This is equivalent to say that $\mathrm{d}\alpha_{T,V} = \mathrm{d}q + \frac{1}{2}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1)$ with

$$q = q(x_1, x_2) = \tfrac{1}{2}(\mu_{11}x_1^2 + (\mu_{12} + \mu_{21})x_1x_2 + \mu_{22}x_2^2).$$

Note that unimodular transformations of variables do not alter the form of the skew-symmetric part of $\alpha_{T,V}$. So, by performing a suitable one it is possible to reduce $q(x_1, x_2)$ to a diagonal form

$$\alpha_{T,V} = \mathrm{d}(\mu y_1^2 + \nu y_2^2) + \tfrac{1}{2}(y_1\,\mathrm{d}y_2 - y_2\,\mathrm{d}y_1).$$

Further, transformations of the form $(y_1, y_2) \to (\lambda y_1, \pm\lambda^{-1} y_2)$ and the possibility to change the sign of $\alpha_{T,V}$ allows to bring it to one of the following canonical forms:

$$\begin{aligned}
&\Psi_\lambda^{\pm}(n)\colon \tfrac{1}{2}\lambda\,\mathrm{d}(z_1^2 \pm z_2^2) + \tfrac{1}{2}(z_1\,\mathrm{d}z - z_2\,\mathrm{d}z_1), \qquad \lambda > 0,\\
&\Psi_1(n)\colon z_1\,\mathrm{d}z_1 + \tfrac{1}{2}(z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1),\\
&\Psi(n)\colon \tfrac{1}{2}(z_1\,\mathrm{d}z_2 - z_2\,\mathrm{d}z_1).
\end{aligned} \tag{6.3}$$

Proposition 6.4. *n-Lie algebras corresponding to the 1-form $\alpha_{T,V}$ of the list (6.3) are mutually non-isomorphic and, therefore, label isomorphic classes of non-unimodular $(n+1)$-dimensional n-Lie algebras.*

Proof. Previous considerations show that any non-unimodular $(n+1)$-dimensional n-Lie algebra is isomorphic to one of the list (6.3). Two algebras of the type $\Psi_\lambda^{\pm}(n)$ corresponding to different λ are not isomorphic since non-vanishing of the skew-symmetric part of $\alpha_{T,V}$ is equivalent to non-unimodularity condition. On the other hand, λ is an invariant of isomorphism type since $2/\lambda$ is equal to the area of a (quasi-) orthonormal base of the symmetric part of $\alpha_{T,V}$ measured by means of its skew-symmetric part. Other types differ by rank or signature of the symmetric part. □

The classification we have got has an interesting *internal* structure. Namely, denote by $B(n)$ the isomorphism type of $(n+1)$-dimensional n-Lie algebras corresponding to the generating polynomial $\frac{1}{2}x_1^2$. Then any $(n+1)$-dimensional algebra can be seen as a "molecule" composed of $B(n)$ and $\Psi(n)$ types of "atoms". More exactly, the above discussion can be resumed as follows

Proposition 6.5. *Any $(n+1)$-dimensional n-Lie algebra can be realized as the sum of mutually compatible algebras each of them being either of type $B(n)$ or of type $\Psi(n)$.*

On the basis of the obtained classification it is not difficult to describe completely the derivation algebras of $(n+1)$-dimensional n-Lie algebras.

A linear operator $A : W \to W$ is called an infinitesimal conformal symmetry of a bilinear form $b(u, v)$ on W if

$$b(Au, v) + b(u, Av) = \mathrm{tr}(A) b(u, v). \tag{6.4}$$

Proposition 6.6. *The Lie algebra of derivations of an $(n+1)$-dimensional n-Lie algebra coincides with the algebra of infinitesimal conformal symmetries of its generating bilinear form.*

Proof. A linear operator A on a linear space can be naturally interpreted as a linear vector field X on it. Moreover, $\mathrm{tr}(A) = \mathrm{div}(X)$. Formula (6.1) for such a field X which is also a symmetry of $\alpha \rfloor V$ reduces to

$$\alpha \rfloor L_X(V) = L_X(\alpha) \rfloor V,$$

which is identical to (6.4). □

We omit a complete description of the derivation algebras which can be easily obtained by applying Proposition 6.5. Just note that inner derivations exhaust all derivations of an $(n+1)$-dimensional algebra iff the rank of its generating form is equal to $n+1$. The following examples illustrate some features of outer derivations.

Example 6.1. Consider the four-dimensional 3-Lie algebra corresponding to the generating polynomial $F = \frac{1}{2}{x_4}^2$. The associated 3-Poisson tensor is

$$P = x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Clearly fields $x_4(\partial/\partial x_1)$, $x_4(\partial/\partial x_2)$, $x_4(\partial/\partial x_3)$ form a basis of inner derivations.

Proposition 6.6 shows that

$$x_4\frac{\partial}{\partial x_4}+x_1\frac{\partial}{\partial x_1},\quad x_4\frac{\partial}{\partial x_4}+x_2\frac{\partial}{\partial x_2},\quad x_4\frac{\partial}{\partial x_4}+x_3\frac{\partial}{\partial x_3}$$

are outer derivations not tangent to the Hamiltonian leaves of P. On the other hand, the following outer derivations

$$x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2},\quad x_2\frac{\partial}{\partial x_2}-x_3\frac{\partial}{\partial x_3},\quad x_3\frac{\partial}{\partial x_3}-x_1\frac{\partial}{\partial x_1}$$

are tangent to these leaves.

Previous method used to get the *n-Bianchi classification* can be extended to inscribe into the n-ary context infinite-dimensional Lie algebras too. This is well illustrated by the following example.

Example 6.2 (Witt algebra). The Witt (or $sl(2,\mathbb{R})$ Kac–Moody) algebra is generated by e_i, $i\in(0,1,2,\dots)$ according to

$$[e_i,e_j]=(j-i)e_{i+j-1}\quad \forall i,j\in\mathbb{N}.$$

Elements e_0,e_1,e_2 generate a three-dimensional subalgebra isomorphic to $sl(2,\mathbb{R})$. It is easy to see that the multiple commutator $\epsilon_k=[e_2,\cdots,[e_2,e_3]](k\ times)$ is equal to $k!e_{3+k}$. So the elements e_0,e_1,e_2,e_3 and ϵ_k, $\forall k\in\mathbb{N}$ constitute a new basis of the Witt algebra.

Let us consider now the Poisson bracket on $\mathbb{R}^3$ given by P_F with

$$P=\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}$$

and

$$F=x_1x_3-x_2^2,$$

i.e

$$P_F=x_1\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}+2x_2\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_3}+x_3\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3}.$$

Then we have the following ordinary Poisson bracket:

$$\{x_1,x_2\}=x_1,\quad \{x_1,x_3\}=2x_2,\quad \{x_2,x_3\}=x_3.$$

So the correspondence

$$[\cdot\,,\,\cdot]\leftrightarrow\{\cdot,\cdot\},\quad e_0\leftrightarrow x_1,\quad e_1\leftrightarrow x_2,\quad e_2\leftrightarrow x_3 \tag{6.5}$$

is an isomorphism of Lie algebras. Moreover, this isomorphism of subalgebras can be extended to an embedding of the whole Witt algebra into the Poisson algebra $\{\cdot,\cdot\}$ according to:

$$[\cdot\,,\,\cdot] \leftrightarrow \{\cdot,\cdot\},\quad e_0 \leftrightarrow x_1,\quad e_1 \leftrightarrow x_2,\quad e_2 \leftrightarrow x_3,$$

$$e_3 \leftrightarrow g = \frac{x_3^2}{x_2}\left(\frac{2F}{x_1x_3} - \frac{x_1x_3}{F} - 1\right),\qquad e_{3+k} \leftrightarrow \frac{1}{k!}\underbrace{\{x_3,\ldots,\{x_3,g\}\}}_{k\ \text{times}}.$$

7. Dynamical aspects

A Hamiltonian vector field $X_{H_1,\ldots,H_{n-1}}$ associated with an n-Poisson structure can be called *n-Poisson*, or *Nambu dynamics*. The corresponding equation of motion is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = X_{H_1,H_2,\ldots,H_{n-1}}f = \{H_1, H_2, \ldots, H_{n-1}, f\}. \tag{7.1}$$

An important peculiarity of a Nambu dynamics is that it admits at least $n-1$ independent constants of motion, namely $H_1, \ldots, H_{n-1}$. Also such a dynamics admits $n-1$ different but mutually compatible Poisson descriptions. The corresponding ith (*usual*) Poisson bracket and Hamiltonian are

$$\{f, g\}_i = \{H_1, \ldots, H_{i-1}, H_{i+1}, \ldots, H_{n-1}, f, g\} \quad \text{and} \quad (-1)^{n-1}H_i,$$

respectively.

So, the fact that a dynamics is a Nambu one can be exploited with the use. Below we give some examples of that.

7.1. The Kepler dynamics

Occasionally, a dynamical vector field Γ admitting $2n-1$ constants of the motion on a $2n$-dimensional manifold M is called *hyper-integrable* or *degenerate*.

If $f_1, f_2, \ldots, f_{2n-1}$ are first integrals for Γ and $f_{2n} \in C^\infty(M)$ is such that $\Gamma(f_{2n}) = 1$, then the $2n$-Poisson bracket

$$\{h_1, h_2, \ldots, h_{2n}\} = \det\left\|\frac{\partial h_i}{\partial f_j}\right\|, \quad i, j \in (1, \ldots, 2n), \tag{7.2}$$

is preserved by Γ which becomes Hamiltonian with respect to (7.2) with the Hamiltonian function $(f_1, f_2, \ldots, f_{2n-1})$.

Of course the corresponding $2n$-Poisson vector is

$$\Lambda = \frac{\partial}{\partial f_1} \wedge \frac{\partial}{\partial f_2} \wedge \cdots \wedge \frac{\partial}{\partial f_{2n}}. \tag{7.3}$$

More generally the $2n$-Poisson bracket

$$\{h_1, h_2, \ldots, h_{2n}\}_F = F\det\left\|\frac{\partial h_i}{\partial f_j}\right\|, \quad i, j \in 1, \ldots, 2n, \tag{7.4}$$

is preserved by Γ iff F is a first integral, i.e. $F = F(f_1, f_2, \ldots, f_{2n-1})$.

The Kepler dynamics illustrates such a situation.

Recall that the Kepler vector field, in spherical–polar coordinates (r, θ, φ) in $\mathbb{R}^3 - \{0\}$, is given by

$$\begin{aligned}\Gamma = \frac{1}{m}\Bigg(& p_r\frac{\partial}{\partial r} + \frac{p_\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{p_\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} \\ & - \frac{1}{r^3}\frac{(p_\theta^2+p_\varphi^2)}{\sin^2\theta}\frac{\partial}{\partial p_r} - \frac{p_\varphi^2\cos\theta}{r^2\sin^3\theta}\frac{\partial}{\partial p_\theta} - \frac{k}{r^2}\frac{\partial}{\partial p_\varphi}\Bigg)\end{aligned} \tag{7.5}$$

with $(p_r, p_\theta, p_\varphi)$ canonical conjugate variables.

Γ is globally Hamiltonian with respect to the symplectic form

$$\omega = \mathrm{d}p_r \wedge \mathrm{d}r + \mathrm{d}p_\theta \wedge \mathrm{d}\theta + \mathrm{d}p_\varphi \wedge d\varphi \tag{7.6}$$

with Hamiltonian H given by (see, for instance [14]):

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) - \frac{k}{r}. \tag{7.7}$$

In action–angle coordinates (J_h, φ_h), $h \in (1, 2, 3)$ (see, for instance, [22]), the Kepler Hamiltonian H, the symplectic form ω and the vector field Γ become:

$$\begin{aligned}&H = -\frac{mk^2}{(J_r + J_\theta + J_\varphi)^2}, \qquad \omega = \sum_h \mathrm{d}J_h \wedge \mathrm{d}\varphi^h, \\ &\Gamma = \nu\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2} + \frac{\partial}{\partial\varphi_3}\right)\end{aligned} \tag{7.8}$$

with $\nu = 2mk^2/(J_r + J_\theta + J_\varphi)^3$.

Functionally independent constants of the motion are: $f_1 = J_1$, $f_2 = J_2$, $f_3 = J_3$, $f_4 = \varphi_1 - \varphi_2$, $f_5 = \varphi_2 - \varphi_3$.

Now it is easy to see that (7.8) becomes 6-Hamiltonian with respect to (7.4) with $F = \nu$. So

$$\{h_1, h_2, h_3, h_4, h_5, h_6\} = \nu\frac{\partial(h_1, h_2, h_3, h_4, h_5, h_6)}{\partial(J_1, J_2, J_3, \varphi_1, \varphi_2, \varphi_3)} \tag{7.9}$$

provides us with a 6-ary bracket for the Kepler dynamics.

In terms of this bracket, the equations of the motion look as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \nu\{J_1, J_2, J_3, \varphi_1 - \varphi_2, \varphi_2 - \varphi_3, f\}. \tag{7.10}$$

By fixing some of the functions h's we get hereditary brackets.

7.2. The spinning particle

Given a dynamics, i.e. a vector field Γ on a manifold M, it colud be interesting to realize it as a Hamiltonian field with respect to a Poisson structure [4]. Below it will be shown how multi-Poisson structures can be used in this connection.

We shall ignore the spatial degree of freedom of the particle and study only the spin variables. Let us treat the spin variables $\mathbf{S} = (S_1, S_2, S_3)$ as elements in $\mathbb{R}^3$. The equations for these variables when the particle interacts with an external magnetic field $\mathbf{B} = (B_1, B_2, B_3)$ are given by

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \mu\epsilon_{ijk}S_j B_k, \tag{7.11}$$

where μ denotes the magnetic moment.

This dynamics has two first integrals, namely, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2$ and $\mathbf{S} \cdot \mathbf{B} = S_1 B_1 + S_2 B_2 + S_3 B_3$ and, in addition, is canonical for the ternary bracket associated with the 3-vector field

$$\frac{\partial}{\partial S_1} \wedge \frac{\partial}{\partial S_2} \wedge \frac{\partial}{\partial S_3}.$$

The most general ternary bracket preserved by dynamics (7.11) is associated with the three vector field

$$f\frac{\partial}{\partial S_1} \wedge \frac{\partial}{\partial S_2} \wedge \frac{\partial}{\partial S_3}, \tag{7.12}$$

where f is a first integral of it.

All Poisson structures obtained by fixing a function $F = F(S^2, \mathbf{S} \cdot \mathbf{B})$ are preserved by the dynamics and are mutually compatible. The corresponding Poisson bracket is

$$\{S_j, S_k\}_F^f = f\epsilon_{jkl}\frac{\partial F}{\partial S_l}.$$

Now we show how the ternary Poisson structure (7.12) allows for the alternative ordinary Poisson brackets described in [4]:

– *Standard description*

$$f = \tfrac{1}{2}, \quad F = S^2.$$

For this choice the algebra generated by the Poisson brackets on linear functions is the $su(2)$ Lie algebra. The Hamiltonian function for the dynamics is the standard one $H = -\mu\mathbf{SB}$.

– *Non-standard description*

Now we take

$$f = \frac{1}{2}, \quad F = S_1^2 + S_2^2 + \frac{1}{2\lambda}\left[\frac{\cosh 2\lambda S_3}{\sinh\lambda} - \frac{1}{\lambda}\right]$$

with Hamiltonian $H = -\mu\lambda S_3$. Here for simplicity we have taken the magnetic field along the third axis. The parameter λ is a deformation parameter and the standard description is recovered for $\lambda \mapsto 0$.

The hereditary Poisson brackets are:

$$\{S_2, S_3\}_F^f = S_1, \quad \{S_1, S_3\}_F^f = S_2, \quad \{S_1, S_2\}_F^f = \frac{1}{2}\frac{\sinh 2\lambda S_3}{\sinh\lambda}.$$

These brackets are a classical realization of the quantum commutation relations for generators of the $U_q(sl(2))$ Hopf algebra.

We also notice that this Poisson Bracket is compatible with the previous one as they are hereditary from the same ternary structure (7.12).

– *Another non-standard description*

There is another choice for f and F which is known to correspond to the classical limit of the $U_q(sl(2))$ Hopf algebra.

It is

$$f = \tfrac{1}{4}\lambda S_3, \quad F = S_1^2 + S_2^2 + S_3^2 + S_3^{-2}.$$

It leads to the following brackets:

$$\{S_2, S_3\}_F^f = \frac{\lambda}{2} S_1 S_3, \quad \{S_1, S_3\}_F^f = \lambda S_2 S_3, \quad \{S_1, S_2\}_F^f = \frac{1}{2}\lambda[S_3^2 - S_3^{-2}].$$

With respect to this Poisson bracket dynamics (7.11) becomes Hamiltonian with Hamiltonian function

$$H = -\frac{2\mu B}{\lambda} \ln S_3$$

with the magnetic field along the third axis.

Of course dynamics (7.11) admits many other Poisson realizations of this type.

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